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DYNAMICS AND BIFURCATION OF $x_{n+1} = \frac{\alpha+\beta x_{n-k}}{A+Bx_n+Cx_{n-k}}, k = 1, 2$

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3. Dr. Alaeddin Elayyan (Member) ....

Dedication

Thankfully I dedicate this thesis to all those who support me and contributed to its success. I dedicate it to my great parents, brothers and sister who through their encouragement and support enabled me to get this degree.
Also I dedicate it to my husband Majdi Raddad for his endless patience , support, care and encouragement.
I am grateful to my supervisor Professor Mohammad Saleh for his support and valuable comments on this thesis.
Many thanks to my teachers, particularly Dr. Marwan Eloqeli and Dr. Alaeddin Elayyan .

THANK YOU ALL

Abstract

The main goal of this thesis is to study the bifurcation of second and third order rational difference equations. We consider the sufficient conditions for the existence of the bifurcation.

We study the second order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots$$

with positive parameters α, β, A, B, C and non-negative initial conditions $\{x_{-k}, x_{-k+1}, \ldots, x_0\}$. We study the dynamic behavior and the direction of the bifurcation of the period-two cycle. Then, we give numerical discussion with figures to support our results.

Also we study the third order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-2}}{A + Bx_n + Cx_{n-2}}, \quad n = 0, 1, 2, \ldots$$

with positive parameters α, β, A, B, C and non-negative initial conditions $\{x_{-k}, x_{-k+1}, \ldots, x_0\}$. We study the dynamic behavior and the direction of the Neimark-Sacker bifurcation. Then, we give numerical discussion with figures to support our results.

Contents

Introduction

In mathematics, a dynamical system is a system in which a function describes the time dependence of a point in a geometrical space. Examples include the mathematical models that describes the flow of water in a pipe, the swinging of a clock pendulum and the number of fish each spring time in a lake.
The study of dynamical system is the focus of dynamical systems theory, which has application to a wide variety of fields such as mathematics, physics, chemistry, biology, medicine, engineering and economics. Dynamical systems are a fundamental part of bifurcation theory which studies the changes in the qualitative or topological structure of systems. A bifurcation occurs when a small change made to the bifurcation parameter of a system causes a qualitative or topological change in its behavior.
This thesis consists of 5 chapters. In chapter 1, we explain in details the definition of dynamical systems and classify them into continuous and discrete systems. Then, we focus on equilibrium points and their stability of first order and higher order discrete dynamical systems. Chapter 2 studies types of bifurcation and their sufficient conditions in simplest forms in discrete dynamical systems of one and two dimensions. Chapter 3 studies the dynamics and behavior of the solutions of the equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{A + Bx_n + Cx_{n-k}}, \quad n = 0, 1, 2, \ldots \qquad (0.1.1)$$

with positive parameters α, β, A, B and C and non-negative initial conditions $\{x_{-k}, x_{-k+1}, \ldots, x_0\}$. We focus on invariant intervals, boundedness of the solutions, periodic solutions of prime period two and global stability of the positive fixed points. Chapter 4 studies the second order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots. \qquad (0.1.2)$$

We focus on the dynamic behavior of the positive fixed point and the type of bifurcation exists where the change of stability occurs . Then, 2 numerical examples are treated to support our results. Chapter 5 studies the third order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-2}}{A + Bx_n + Cx_{n-2}}, \quad n = 0, 1, 2, \ldots. \tag{0.1.3}$$

We also focus on the stability of the positive fixed point and the type of bifurcation exists where the change of stability occurs. Then, we give numerical discussion with figures to support our results.

Equation (0.1.1) was studied by Guo-Mei Tang, Lin-Xia Hu, and Gang Ma in [5]. Local and global stability, period two solutions, boundedness, invariant intervals and semicycles were studied.

Equation (0.1.2) was studied by Lin-Xia Hu, Wan-Tong Li, Hong-Wu Xu in [6]. Boundedness, invariant intervals, semicycles and global stability of the positive fixed point was investigated. Also this equation was studied by Ladas in [7].

Equation (0.1.2) and equation (0.1.3) were studied by Ladas in [2].

The aim of this thesis is to study the bifurcation of the second order rational difference equation (0.1.2) and the third rational difference equation (0.1.3).

1

Dynamical System

1.1 Introduction

The dynamical system consists of phase (state) space which consists of points that represent all possible states of the system and a low of evolution of the state in time. Every point in the state space must describe the current position of the system and determine its evolution. Dynamical system is important in physics, biology, chemistry, economics, social science and many disciplines.
In this chapter we review some basic preliminaries.
Evolution law always defined as a map f^t for given $t \in T$ and this map defined on the state space X as follows

$$f^t : X \to X$$

f^t is called evolution operator of the dynamical system which transforms an initial state x_0 into some state x_t at time t where x_t denotes $f^t x_0$.
Also we use $x(t)$ to denote x_t or $f^t x_0$.

Definition 1. *[1] A dynamical system is a triple $\{T, X, f^t\}$, where T is a time set, X is a state space and $f^t : X \to X$ is a family of evolution operators parameterized by $t \in T$.*

Dynamical systems are two types:

- systems with continuous (real) time $T = \mathbb{R}^1$ which called continuous-time dynamical systems and the law of evolution is differential equation.
- systems with discrete (integer) time $T = \mathbb{Z}$ which called discrete-time dynamical systems and the law of evolution is difference equation.

Example 1.1. *[1] Consider the plane $X = \mathbb{R}^2$ and a family of linear nonsingular transformations on X given by the matrix depending on $t \in \mathbb{R}$*

$$f^t = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix}$$

where λ, $\mu \neq 0$ are real numbers. Obviously, it specifies a continuous-time dynamical system on X. The system is defined for all (x, t) and the map f^t is continuous in x, as well as in t.$\diamond$

Example 1.2. *[1] Take the space $X = \Omega_2$ of all bi-infinite sequences of two symbols $\{1, 2\}$. Consider a map $\sigma : X \to X$, which transforms the sequence*

$$\omega = \{\ldots, \omega_{-2}, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots\} \in X$$

into the sequence $\theta = \sigma(\omega)$,

$$\theta = \{\ldots, \theta_{-2}, \theta_{-1}, \theta_0, \theta_1, \theta_2, \ldots\} \in X$$

where

$$\theta_k = \omega_{k+1}, k \in \mathbb{Z}$$

The map σ merely shifts the sequence by one position to the left. It is called a shift map. The shift map defines a discrete-time dynamical system on X, $f^k = \sigma^k$. Notice that two sequences, θ, ω are equivalent if and only if $\theta = \sigma^{k_0}(\omega)$ for some $k_0 \in \mathbb{Z}$.
$\diamond$

In this chapter we consider the discrete-time dynamical systems only .

1.2 Dynamics Of First-order difference equations

Consider the function $f : X \to X$ and the first-order difference equation of the form

$$x_{t+1} = f(x_t) \tag{1.2.1}$$

where x_0 is an initial condition. Note that

$$x_1 = f(x_0)$$

,

$$x_2 = f(x_1) = f(f(x_0))$$
$$\vdots$$
$$x_{t+1} = f(x_t) = \ldots = f^{t+1}(x_0)$$

Definition 2. *[3] A point $\bar{x} \in X$ is called an equilibrium (fixed) point of equation (1.2.1) if $f(\bar{x}) = \bar{x}$.*

In example (1.2), we have only the following two fixed points

$$\omega^1 = \{\ldots, 1, 1, 1, \ldots\}$$

and

$$\omega^2 = \{\ldots, 2, 2, 2, \ldots\}.$$

Note that the fixed point $\bar{x}$ of the equation $x_t = f^t x$ is a constant solution, since if $x_0 = \bar{x}$ is an initial point, then $x_1 = f(\bar{x}) = \bar{x}$ and $x_2 = f(x_1) = f(\bar{x}) = \bar{x}$ and so on.
We can find the equilibrium points of any map f graphically. We draw a graph of f in $(x_n, x_{n+1})-$ plane. Next, we find the points where the map f intersect the diagonal line $y = x$. X-coordinate of those points are the equilibrium points of the map f.

Example 1.3. *Consider the function*

$$f(x) = 3x - x^2.$$

Fixed points of the function are the roots of the function $h(x) = f(x) - x = 2x - x^2$, this implies that the fixed points are $\bar{x}_1 = 0$ and $\bar{x}_2 = 2$.
To find these fixed points graphically, we use figure (1.1).

Equilibrium points are two types: hyperbolic and nonhyperbolic. An equilibrium point $\bar{x}$ of a map f is said to be hyperbolic if $\mid \acute{f}(\bar{x}) \mid \neq 1$. Otherwise, it is nonhyperbolic.

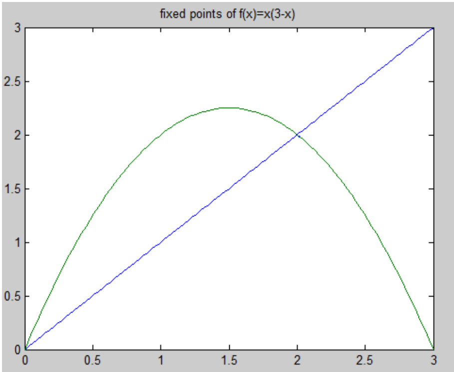

Fig. 1.1: The fixed points of $f(x) = 3x - x^2$ are the intersection points with the diagonal line.

Definition 3. *[1] An orbit starting at x_0 is an ordered subset of the state space X such that $Or(x_0) = \{x \in X : x = f^t x_0, \text{ for all } t \in T \text{ such that } f^t x_0 \text{ is defined}\}$.*

It is possible to have a solution which is not an equilibrium point but reaches an equilibrium point after finitely many iterations. This leads to the definition of eventually equilibrium (fixed) point.

Definition 4. *[3] Consider equation (1.2.1). Let x^* be a point in the domain of f. If there exists a positive integer r and an equilibrium point $\bar{x}$ of f such that $f^r(x^*) = \bar{x}$ and $f^{r-1} \neq \bar{x}$, then x^* is an eventually equilibrium point.*

Definition 5. *[3] Let x^* be in the domain of f. If for some positive integer k $f^k(x^*) = x^*$, then x^* is called a k-periodic point of f. The periodic orbit of x^*, $O(x^*) = \{x^*, f(x^*), f^2(x^*), \ldots, f^{k-1}(x^*)\}$ is called a k-cycle.*

Stability Of One-Dimensional Maps

Behavior of the solutions near the equilibrium points is one of the main objectives in the study of any dynamical system. We introduce the definition of stability and its types.

Definition 6. *[3] Consider equation (1.2.1).*

1. *The equilibrium point $\bar{x}$ is stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that $\mid x_0 - \bar{x} \mid < \delta$ implies $\mid f^n(x_0) - \bar{x} \mid < \epsilon$ for all $n > 0$. If $\bar{x}$ is not stable, then it is called unstable.*

2. *The equilibrium point $\bar{x}$ is said to be attracting if there exists $\eta > 0$ such that $\mid x_0 - \bar{x} \mid < \eta$ implies $lim_{n \to \infty} x_n = \bar{x}$.*
   If $\eta = \infty$, $\bar{x}$ is called a global attractor or globally attracting.

3. *The equilibrium point $\bar{x}$ is an asymptotically stable if it is stable and attracting. If $\eta = \infty$, $\bar{x}$ is said to be globally asymptotically stable.*

We have simple criterion for the asymptotic stability of equilibrium points

Theorem 1.1. *[3] Let $\bar{x}$ be an equilibrium point of the difference equation (1.2.1) where f is continuously differentiable at $\bar{x}$. Then the following statements hold true:*

1. *If $\mid f'(\bar{x}) \mid < 1$, then $\bar{x}$ is asymptotically stable.*

2. *If $\mid f'(\bar{x}) \mid > 1$, then $\bar{x}$ is unstable.* $\diamond$

But what if we have a non-hyperbolic fixed point? The next theorems will give the answer.

Theorem 1.2. *[3] Suppose that for an equilibrium point $\bar{x}$ of (1.2.1) $f'(\bar{x}) = 1$. The following statements then hold:*

- *If $f''(\bar{x}) \neq 0$, then $\bar{x}$ is unstable.*

- *If $f''(\bar{x}) = 0$ and $f'''(\bar{x}) > 0$, then $\bar{x}$ is unstable.*

- *If* $f''(\bar{x}) = 0$ *and* $f'''(\bar{x}) < 0$, *then* $\bar{x}$ *is asymptotically stable.* $\diamond$

Definition 7. *[4] The Schwarzian derivative, Sf, of a function f is defined by*

$$Sf = \frac{f'''(x)}{f'(x)} - \frac{3}{2}[\frac{f''(x)}{f'(x)}]^2.$$

And if $\acute{f}(\bar{x}) = -1$, *then*

$$Sf(\bar{x}) = -f'''(\bar{x}) - \frac{3}{2}[f''(\bar{x})]^2.$$

Theorem 1.3. *[3] Suppose that for the equilibrium point* $\bar{x}$ *of (1.2.1),* $f'(\bar{x}) = -1$, *then the following statements hold:*

1. *If* $Sf(\bar{x}) < 0$, *then* $\bar{x}$ *is asymptotically stable.*
2. *If* $Sf(\bar{x}) > 0$, *then* $\bar{x}$ *is unstable.*

Example 1.4. *Consider the map*

$$f(x) = 3x - x^2, \quad x \in [0, 3]$$

Find the equilibrium points and determine their stability.
The fixed points of f are the roots of the function

$$h(x) = f(x) - x = 2x - x^2.$$

We have two fixed points $\bar{x}_1 = 0$ *and* $\bar{x}_2 = 2$.
Note that $f'(x) = 3 - 2x$.

$$f'(0) = 3 > 1,$$

$$f'(2) = -1$$

and

$$Sf(2) = -6 < 0$$

Theorem (1.1) implies that $\bar{x}_1$ *is unstable fixed point and theorem (1.3) implies that* $\bar{x}_2$ *is stable fixed point.*

Also we have graphical techniques for analyzing the stability of equilibrium points for (1.2.1). Cobweb diagram is an important one. We draw the curve $x_{n+1} = f(x_n)$ in the (x_n, x_{n+1})-plane. From an initial point x_0, we know the value x_1 by drawing a vertical line through x_0 which intersects the graph of f at (x_0, x_1). Next draw a horizontal line from (x_0, x_1), this line intersects the diagonal line $y = x$ at the point (x_1, x_1). Next draw a vertical line from the point (x_1, x_1), this vertical line intersects the graph of f at the point (x_1, x_2). Continue at the same way, you can find $x(n)$ for all $n > 0$.

Example 1.5. *Consider the function in example (1.4). Cobweb diagram (1.2) shows that $\bar{x}_2$ is stable.*

Figure (1.3) shows the behavior of x_n near the fixed point $\bar{x}_2$.

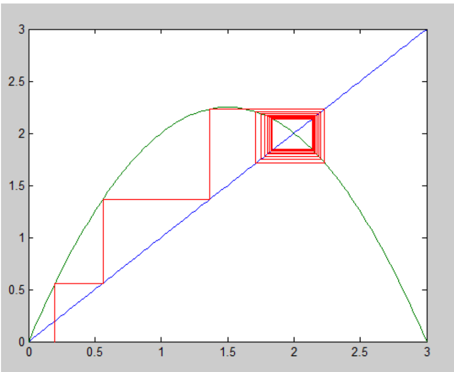

Fig. 1.2: The Cobweb diagram: $\bar{x}_2$ is asymptotically stable.

1.3 Dynamics Of Higher Order Difference Equation

In this section we deal with higher order difference equations.

Definition 8. *[5] Let I be an interval of real numbers and let*

$$f : I^k \to I$$

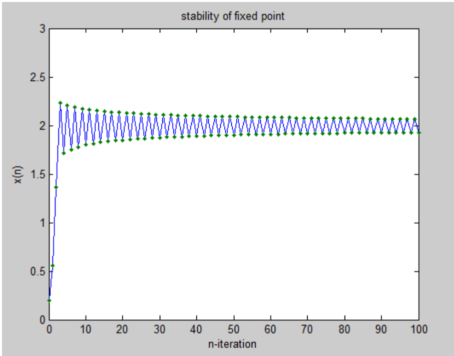

Fig. 1.3: The behavior of the solutions near the fixed point x=1.

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, \dots, x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), n = 0, 1, \dots \tag{1.3.1}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.
The equilibrium point $\bar{x}$ of the equation (1.3.1) is the point that satisfies $\bar{x} = f(\bar{x}, \bar{x}, \dots, \bar{x})$.

Definition 9. *The solution $\{x_n\}_{n=-k}^{\infty}$ of the difference equation $x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k}), n = 0, 1, 2, \dots$ is periodic if there exists a positive integer r such that $x_{n+r} = x_n$. The smallest such r is called the prime period of the solution of the difference equation.*

Now we introduce the definition of stability of equilibrium points of higher order difference equations.

Definition 10. *[7] Let $\bar{x}$ be an equilibrium point of (1.3.1), then:*

1. *The equilibrium point $\bar{x}$ is called locally stable if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_0 \in I$ with $\mid x_{-k} - \bar{x} \mid + \mid x_{-k+1} - \bar{x} \mid + \dots + \mid x_0 - \bar{x} \mid < \delta$, we have $\mid x_n - \bar{x} \mid < \epsilon$ for all $n \geq k$.*

2. *$\bar{x}$ is called locally asymptotically stable if it is stable and there exists $\mu > 0$ such that for all $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$ with $\mid x_{-k} - \bar{x} \mid + \mid x_{-k+1} - \bar{x} \mid + \ldots + \mid x_0 - \bar{x} \mid < \mu$ we have $lim_{n\to\infty} x_n = \bar{x}$.*

3. *$\bar{x}$ is called a global attractor if for all $x_{-k}, x_{-k+1}, \ldots, x_0 \in I$ we have $lim_{n\to\infty} x_n = \bar{x}$.*

4. *$\bar{x}$ is called globally asymptotically stable if it is asymptotically stable and global attractor.*

5. *$\bar{x}$ is called unstable if it is not stable.*

We have several theorems that we can use to determine the stability of the fixed points of high order difference equations.

Definition 11. *[5] Consider the difference equation (1.3.1). Then the linearized equation associated with this difference equation is*

$$x_{n+1} = \sum_{j=0}^{k} \frac{\partial f}{\partial u_j}(\bar{x}, \bar{x}, \ldots, \bar{x}) x_{n-j}, n = 0, 1, 2, \ldots$$

and the characteristic equation is

$$\lambda^{k+1} = \sum_{j=0}^{k} \frac{\partial f}{\partial u_j}(\bar{x}, \bar{x}, \ldots, \bar{x}) \lambda^{k-j}, n = 0, 1, 2, \ldots$$

.

Theorem 1.4. *[3]**[The Linearized Stability Result]** Suppose f is continuously differentiable function on an open neighborhood $G \subset R^{k+1}$ of $(\bar{x}, \bar{x}, \ldots, \bar{x})$, where $\bar{x}$ is a fixed point of (1.3.1), then the following statements are true:*

1. *If all the characteristic roots of the characteristic equation of the linearized equation around $\bar{x}$ lie inside the unit disc in the complex plane, then the equilibrium point $\bar{x}$ is locally asymptotically stable.*

2. *If at least one of these roots is outside the unit disc, then the equilibrium point $\bar{x}$ is unstable.*

3. *If one root is on the unit disc and all the other roots are either inside or on the unit disc, then the equilibrium point $\bar{x}$ may be stable or unstable.* $\diamond$

For our equations it is easier to use the following theorems to determine the stability of the positive equilibrium points.

Theorem 1.5. *[3] Consider the difference equation*

$$x(n+2) + px(n+1) + qx(n) = 0, n = 0, 1, \dots . \tag{1.3.2}$$

The characteristic polynomial is

$$p(\lambda) = \lambda^2 + p\lambda + q.$$

Then the zero solution is asymptotically stable if and only if

$$\mid p \mid < 1 + q < 2. \qquad \diamond$$

We can convert equation (1.3.2) to second-dimensional system by the following: Let

$$x_1(n) = x(n-1),$$
$$x_2(n) = x(n).$$

We have the system

$$x_1(n+1) = x_2(n)$$
$$x_2(n+1) = px_2(n) + qx_1(n). \tag{1.3.3}$$

Let $X(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}$ and let $f(X(n)) = \begin{pmatrix} f_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} x_2(n) \\ px_2(n) + qx_1(n) \end{pmatrix}$.
Then system (1.3.3) is equivalent to

$$X(n+1) = f(X(n))$$

Let $A = Jf(\bar{x})$ is the Jacobian matrix of f evaluated at the fixed point $\bar{x}$ where

$$Jf(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} |_{\bar{x}}$$

Theorem 1.6. *[4] Consider the map $f : G \subset \mathbb{R}^2 \to \mathbb{R}^2$ ba a C^1 map, where G is an open subset of $\mathbb{R}^2$, $\bar{x}$ is a fixed point of f, $A = Jf(\bar{x})$ and $\rho(A)$ is the spectral norm of A where $\rho(A) = max_i\{\mid \lambda_i \mid, \lambda_i$ are the eigenvalues of $A\}$. Then the following statement hold true:*

1. *If $\rho(A) < 1$, then $\bar{x}$ is asymptotically stable.*
2. *If $\rho(A) > 1$, then $\bar{x}$ is unstable.*
3. *If $\rho(A) = 1$, then $\bar{x}$ may or may not be stable.$\diamond$*

One can use the following theorem to determine the stability of the fixed points of equation (1.3.2).

Theorem 1.7. *[4] Consider the map*

$$x \to f(x), \quad x \in \mathbb{R}^2$$

and let $A = Jf(\bar{x})$. Then $\rho(A) < 1$ if and only if

$$\mid trA \mid -1 < \det A < 1$$

where trA and $\det A$ denote trace and determinant of the matrix A respectively.$\diamond$

If equations (1.3.2) has a parameter $\alpha \in \mathbb{R}^m$, then we write them as a second dimensional system and then it can be denoted by

$$X(n+1) = f(X(n), \alpha), \qquad X(n) \in \mathbb{R}^2, \alpha \in \mathbb{R}^m. \tag{1.3.4}$$

One can use the trace and determinant of the Jacobian matrix of f $A = Jf(\bar{x}, \bar{\alpha})$ to study the stability of the fixed points $(\bar{x}, \bar{\alpha})$ of equation (1.3.4). By theorem (1.7) we can determine the values of the parameters where the change in the phase portrait occurs. The lines: $detA = -trA - 1$, $detA = trA - 1$ and $detA = 1$ enclosed the stabile region in the trace-determinant plane. Also those lines are important in the study of the bifurcation of any second-dimensional system.

Theorem 1.8. *[3] For the third-order difference equation*

$$x(n+3) + p_1 x(n+2) + p_2 x(n+1) + p_3 x(n) = 0, \tag{1.3.5}$$

the characteristic polynomial is

$$p(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3.$$

A necessary and sufficient condition for the zero solution to be asymptotically stable is

$$\mid p_1 + p_3 \mid < 1 + p_2 \quad \textit{and} \quad \mid p_2 - p_1 p_3 \mid < 1 - p_3^2. \qquad \diamond$$

2

Bifurcation Of Fixed Points

Bifurcation is a general term. Its use is to describe the orbit structure near non-hyperbolic fixed points.

Definition 12. *Consider a dynamical system that depends on parameters*

$$x \to f(x, \alpha)$$

where $x \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}^m$ *represent phase variables and parameters respectively. As the parameters varies, the phase portrait also varies. Bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation value.*

Bifurcation diagram display the location and stability of fixed point as a function of the parameter in a single plot. The locations of unstable fixed points are shown dashed ,while stable fixed points are represented by solid lines.
In one-dimensional systems, there are four types of bifurcations known as saddle-node, transcritical, pitchfork and period-doubling bifurcation.

2.1 Bifurcation Of one-parameter family of one-dimensional maps

A fixed point (x^*, α^*) of a one-dimensional map is a bifurcation point if either only one branch or more than one branch of fixed points passes through (x^*, α^*) in the $\alpha - x$ plane, then it lies entirely on one side of the line $\alpha = \alpha^*$ in the $\alpha - x$ plane.
In this section we present general conditions under which a one-parameter family of one-dimensional map will undergo a saddle-node, pitchfork, transcritical and period-doubling bifurcation.
Remark: If we have more parameters in the problem, we will consider all, except one, as fixed.

2.1.1 The Saddle-node Bifurcation

Saddle-node bifurcation associated with the appearance of a slope of 1. A unique curve of fixed points passes through the non hyperbolic fixed point (x^*, α^*). Moreover, the curve lay entirely on one side of the $\alpha = \alpha^*$ in the $x - \alpha$ plane.

Theorem 2.1. ***[The Saddle-Node Bifurcation]****[4] Suppose that $f(x, \alpha)$ is a C^2 one parameter family of one-dimensional maps (i.e both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial \alpha^2}$ exist and are continuous), and x^* is a fixed point of $f(x, \alpha^*)$, with $\frac{\partial f}{\partial x}(x^*, \alpha^*) = 1$. Assume further that*

$$A = \frac{\partial f}{\partial \alpha}(x^*, \alpha^*) \neq 0$$

And

$$B = \frac{\partial^2 f}{\partial x^2}(x^*, \alpha^*) \neq 0$$

Then there exists an interval I around x^* and a C^2 map $\alpha = p(x)$, where $p : I \to \mathbb{R}$ such that $p(x^*) = \alpha^*$ and $f(x, p(x)) = x$. Moreover, if $AB < 0$, then the fixed points exist for $\alpha > \alpha^*$, and, if $AB < 0$, then the fixed points exist for $\alpha < \alpha^*$.

We need the following theorem to proof theorem (2.1).

Theorem 2.2. ***[The Implicit Function Theorem]****[4] Suppose that $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1 map in both variables such that for some $(\mu^*, x^*) \in \mathbb{R} \times \mathbb{R}, G(x^*, \mu^*) = 0$ and $\frac{\partial G}{\partial \mu}(x^*, \mu^*) \neq 0$. Then, there exists an open interval J around μ^*, an open interval I around x^* and a C^1 map $\mu = p(x)$, where $p : I \to J$ such that*

1. $p(x^*) = \mu^*$
2. $G(x, p(x)) = 0$, *for all* $x \in I$ $\diamond$

Proof of theorem (2.1): Suppose that $f(x, \alpha)$ is a C^2 one-parameter of one -dimensional map and x^* is a fixed point of $f(x, \alpha^*)$ with $\frac{\partial f}{\partial x}(x^*, \alpha^*) = 1$. Also suppose $A = \frac{\partial f}{\partial \alpha}(x^*, \alpha^*) \neq 0$ and $B = \frac{\partial^2 f}{\partial x^2}(x^*, \alpha^*) \neq 0$.
Let

$$G(x, \alpha) = f(x, \alpha) - x.$$

Note that G is a C^1 map where

$$G(x^*, \alpha^*) = 0$$

and

$$\frac{\partial G}{\partial \alpha}(x^*, \alpha^*) = \frac{\partial f}{\partial \alpha}(x^*, \alpha^*) = A \neq 0.$$

By implicit function theorem there is an open interval J around x^* and an open interval I around α^* and a C^1 map $\alpha = p(x)$, where $p : J \to I$ such that

$$p(x^*) = \alpha^*,$$

and

$$G(x, p(x)) = 0, \forall x \in J.$$

Thus

$$f(x, p(x)) = x, \forall x \in J. \tag{2.1.1}$$

Differentiate both sides of equation (2.1.1) with respect to x and then substitute (x^*, α^*), we obtain

$$\frac{\partial f}{\partial x}(x^*, \alpha^*) + \frac{\partial f}{\partial \alpha}(x^*, \alpha^*)p'(x^*) = 1.$$

Since we assume $\frac{\partial f}{\partial x}(x^*, \alpha^*) = 1$ and $\frac{\partial f}{\partial \alpha}(x^*, \alpha^*) \neq 0$, $p'(x^*) = 0$.

Differentiate (2.1.1) again with respect to x and then substitute (x^*, α^*), we obtain

$$\frac{\partial^2 f}{\partial x^2}(x^*, \alpha^*) + \frac{\partial f}{\partial \alpha}(x^*, \alpha^*)p''(x^*) + \frac{\partial^2 f}{\partial \alpha^2}(x^*, \alpha^*)p'(x^*) = 0.$$

Substitute $p'(x^*) = 0$, we have

$$\frac{\partial^2 f}{\partial x^2}(x^*, \alpha^*) + \frac{\partial f}{\partial \alpha}(x^*, \alpha^*)p''(x^*) = 0.$$

This implies

$$p''(x^*) = -\frac{\frac{\partial f}{\partial \alpha}(x^*, \alpha^*)}{\frac{\partial^2 f}{\partial x^2}(x^*, \alpha^*)} = -\frac{B}{A}.$$

Note that the function $p(x) = \alpha$, which represents the fixed points of $f(x, \alpha)$, has critical point at $x = x^*$. If $AB < 0$, then the the curve $p(x)$ is concave upward at $x = x^*$ and if $AB > 0$, then $p(x)$ concave downward at $x = x^*$. $\diamond$

Example 2.1. *Consider the map*

$$f(x,\alpha) = \alpha - x^2, \quad x \in \mathbb{R}, \alpha \in \mathbb{R}$$

Fixed points of the map $f(x,\alpha)$ are given by the equation

$$f(x,\alpha) - x = \alpha - x^2 - x = 0$$

or

$$x = -\frac{1 \pm \sqrt{1+4\alpha}}{2}.$$

Note that fixed points are exist for $\alpha \geq -\frac{1}{4}$. At $\alpha^* = -\frac{1}{4}$ we have the fixed point $x^* = -\frac{1}{2}$. This fixed point is non-hyperbolic since $\frac{\partial f}{\partial x}(x^*,\alpha^*) = 1$.
Observe that $\frac{\partial f}{\partial \alpha}(x^*,\alpha^*) = 1 \neq 0$ and $\frac{\partial^2 f}{\partial x^2}(x^*,\alpha^*) = -2 \neq 0$. By theorem (2.1), the saddle-node bifurcation is present at $(-\frac{1}{2}, -\frac{1}{4})$.
In order to draw the bifurcation diagram we check the stability of the system near the bifurcation point $(-\frac{1}{2}, -\frac{1}{4})$. Note that $\acute{f}(x,\alpha) = -2x$. The upper branch $x = -\frac{1-\sqrt{1+4\alpha}}{2}$ is asymptotically stable if

$$\mid \acute{f}(-\frac{1-\sqrt{1+4\alpha}}{2}, \alpha) \mid < 1. \tag{2.1.2}$$

Inequality (2.1.2) holds if $-\frac{1}{4} < \alpha < \frac{3}{4}$. So the upper branch is stable when $-\frac{1}{4} < \alpha < \frac{3}{4}$. The lower branch is unstable since $\mid \acute{f}(-\frac{1+\sqrt{1+4\alpha}}{2}, \alpha) \mid = 1 + \sqrt{1+4\alpha} > 1$ for all values of α. See figure (2.1)

2.1.2 Transcritical Bifurcation

Consider the one-dimensional map

$$x \to f(x,\alpha), \quad x \in \mathbb{R}, \alpha \in \mathbb{R}$$

with x^* as a fixed point of $f(x,\alpha^*)$.
Transcritical bifurcation is another type of bifurcation in one-dimensional systems. This type appears when we have two curves of fixed points intersected at the non

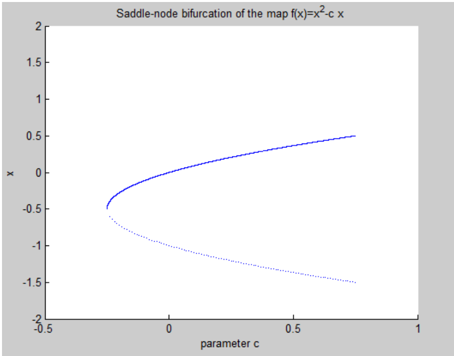

Fig. 2.1: Saddle-node bifurcation.

hyperbolic fixed point (x^*, α^*) in the $\alpha - x$ plane. Both curves existed on either sides of the line $\alpha = \alpha^*$. However, the stability of the fixed point along a given curve changed on passing through $\alpha = \alpha^*$.

Theorem 2.3. *[9] Suppose that $f(x, \alpha)$ is a $C^r(r \geq 2)$ map where $x \in \mathbb{R}, \alpha \in \mathbb{R}$ and (x^*, α^*) is a non-hyperbolic fixed point of $f(x, \alpha)$ such that*

$$\frac{\partial f}{\partial x}(x^*, \alpha^*) = 1,$$

$$\frac{\partial f}{\partial \alpha}(x^*, \alpha^*) = 0,$$

$$\frac{\partial^2 f}{\partial x \partial \alpha}(x^*, \alpha^*) \neq 0,$$

and

$$\frac{\partial^2 f}{\partial x^2}(x^*, \alpha^*) \neq 0.$$

Then f undergoes a transcritical bifurcation at (x^*, α^*).

Proof: Consider the map $f(x, \alpha)$ with a fixed point (x^*, α^*) which satisfies the hypotheses. Let

$$G(x, \alpha) = f(x, \alpha) - x$$

since we assume that $\frac{\partial f}{\partial \alpha}(x^*, \alpha^*) = 0$,

$$\frac{\partial G}{\partial \alpha}(x^*, \alpha^*) = \frac{\partial f}{\partial \alpha}(x^*, \alpha^*) = 0$$

so we can not apply the Implicit Function Theorem. Let

$$B(x, \alpha) = \begin{cases} \frac{G(x,\alpha)}{x-x^*}, & \text{if } x \neq x^*; \\ \frac{\partial G}{\partial x}(x^*, \alpha), & \text{if } x = x^* \end{cases}$$

note that

$$B(x^*, \alpha^*) = \frac{\partial G}{\partial x}(x^*, \alpha^*) = \frac{\partial f}{\partial x}(x^*, \alpha^*) - 1 = 0$$

and

$$\frac{\partial B}{\partial \alpha}(x^*, \alpha^*) = \frac{\partial}{\partial \alpha}(\frac{\partial G}{\partial x}(x^*, \alpha^*)) = \frac{\partial^2 f}{\partial \alpha \partial x}(x^*, \alpha^*) \neq 0.$$

By Implicit Function Theorem there is a C^1 map $\alpha = p(x)$ defined on an interval I around x^* such that $p(x^*) = \alpha^*$ and

$$B(x, p(x)) = 0, \quad \forall x \in I. \tag{2.1.3}$$

Hence, $\frac{G(x,p(x))}{x-x^*} = 0$ for $x \neq x^*$ and so $f(x, p(x)) = x$. Differentiate (2.1.3) with respect to x and then substitute (x^*, α^*), we have

$$\frac{\partial B}{\partial x}(x^*, \alpha^*) + \frac{\partial B}{\partial \alpha}(x^*, \alpha^*)\acute{p}(x^*) = 0.$$

Since

$$\frac{\partial B}{\partial x}(x^*, \alpha^*) = \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(x^*, \alpha^*) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x^*, \alpha^*) \neq 0$$

and

$$\frac{\partial B}{\partial \alpha}(x^*, \alpha^*) = \frac{\partial}{\partial \alpha}(\frac{\partial G}{\partial x}(x^*, \alpha^*)) \neq 0,$$

we have $p'(x^*) \neq 0$. This means that $p(x)$ does not coincide with $x = x^*$and exists on both sides of $\alpha = \alpha^*$. $\diamond$

Example 2.2. *Consider the map*

$$f(x, \alpha) = x^2 + \alpha x, \quad x \in \mathbb{R}, \alpha \in \mathbb{R}.$$

The fixed points of the map are the roots of the function

$$h(x,\alpha) = f(x,\alpha) - x = x^2 + \alpha x - x.$$

Hence, $f(x,\alpha)$ *has two fixed points* $x_1 = 0$ *for any value of* α *and* $x_2 = 1-\alpha$ *for* $\alpha \neq 0$. *Since*

$$\frac{\partial f}{\partial x}(0,1) = 1,$$

$(0,1)$ *is a non-hyperbolic fixed point of the map* f.
To check the stability of the fixed points near the point $(0,1)$ *we find when*

$$\mid \frac{\partial f}{\partial x}(0,\alpha) \mid < 1 \tag{2.1.4}$$

and

$$\mid \frac{\partial f}{\partial x}(1-\alpha,\alpha) \mid < 1. \tag{2.1.5}$$

Inequality (2.1.4) holds if $-1 < \alpha < 1$ *and inequality (2.1.5) holds if* $1 < \alpha < 3$. *So the branch* $x = 0$ *is asymptotically stable if* $-1 < \alpha < 1$ *and the branch* $x = \alpha - 1$ *is asymptotically stable if* $1 < \alpha < 3$. *Note that the two branches intersects at the bifurcation point* $(0,1)$ *where the branch* $x = 0$ *is stable and the other branch* $x = \alpha - 1$ *is unstable before* $(0,1)$. *Beyond* $\alpha = 1$, *the branch* $x = 0$ *becomes unstable and the other branch* $x = \alpha - 1$ *becomes stable. So change of stability occurs at* $\alpha = 1$.

2.1.3 Pitchfork Bifurcation

Consider the one-dimensional map

$$x \to f(x,\alpha), \quad x \in \mathbb{R}, \alpha \in \mathbb{R}$$

with x^* as a non fixed point of $f(x,\alpha^*)$.
Pitchfork bifurcation is a type of bifurcation in one-dimensional systems which appears when we have two curves of fixed points intersected at the non hyperbolic fixed point (x^*,α^*) in the $\alpha - x$ plane. Only one curve existed on both sides of $\alpha = \alpha^*$; however, its stability changes on passing through $\alpha = \alpha^*$. The other curve

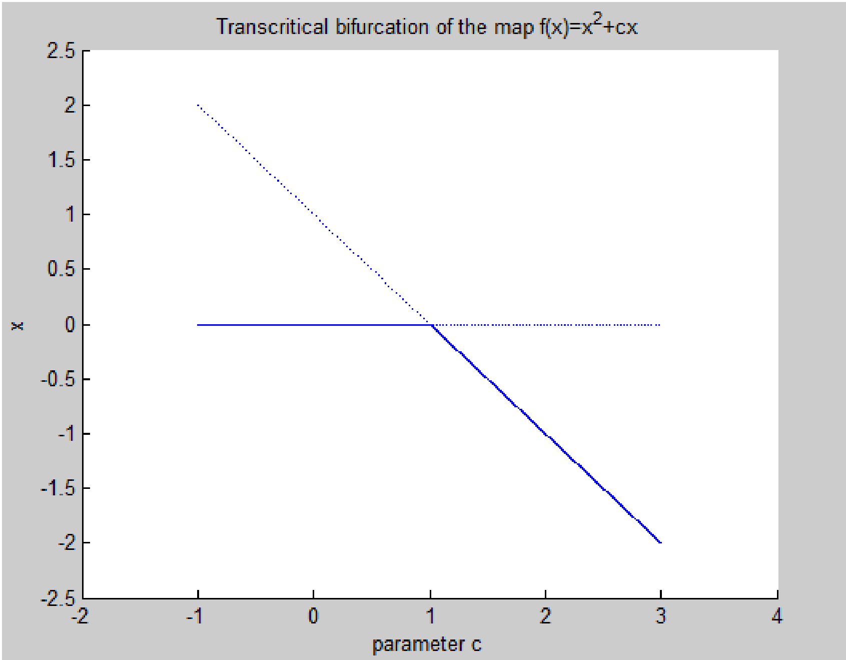

Fig. 2.2: Transcritical bifurcation of $f(x, \alpha) = x^2 + \alpha x$.

of fixed points lay entirely to one side of the line $\alpha = \alpha^*$ and has a stability type that is the opposite of the other curve.

Theorem 2.4. *[9] Suppose that $f(x, \alpha)$ is a C^2 one-parameter family of one-dimensional map, where $x \in \mathbb{R}, \alpha \in \mathbb{R}$ with a non-hyperbolic fixed point (x^*, α^*) such that*

$$f(x^*, \alpha^*) = x^* \quad \text{and} \quad \frac{\partial f}{\partial x}(x^*, \alpha^*) = 1.$$

Assume further that

$$A = \frac{\partial f}{\partial \alpha}(x^*, \alpha^*) = 0,$$
$$C = \frac{\partial^2 f}{\partial x^2}(x^*, \alpha^*) = 0,$$
$$D = \frac{\partial^2 f}{\partial x \partial \alpha}(x^*, \alpha^*) \neq 0$$

and

$$E = \frac{\partial^3 f}{\partial x^3}(x^*, \alpha^*) \neq 0.$$

Then pitchfork bifurcation is present at (x^*, α^*).

Proof: Consider the map $f(x,\alpha)$ with a fixed point (x^*,α^*) which satisfies the hypotheses. The fixed points of this map are represented by the equation

$$G(x,\alpha) = f(x,\alpha) - x = 0.$$

Note that $G(x^*,\alpha^*) = 0$. Observe that $\frac{\partial G}{\partial x}(x^*,\alpha^*) = 0$ so it is not possible to apply the Implicit Function Theorem. Let

$$B(x,\alpha) = \begin{cases} \frac{G(x,\alpha)}{x-x^*}, & if x \neq x^*; \\ \frac{\partial G}{\partial x}(x^*,\alpha), & if x = x^* \end{cases}$$

Note that $B(x^*,\alpha^*) = 0$ and

$$\frac{\partial B}{\partial \alpha}(x^*,\alpha^*) = \frac{\partial}{\partial \alpha}[\frac{\partial G}{\partial x}(x^*,\alpha^*)] = \frac{\partial^2 f}{\partial x \partial \alpha}(x^*,\alpha^*) \neq 0.$$

By the Implicit Function Theorem, there is a map $\alpha = p(x)$ defined on an interval I around x^* such that $\alpha^* = f(x^*)$ and

$$B(x,p(x)) = 0, \forall x \in I. \tag{2.1.6}$$

Differentiate equation (2.1.6) with respect to x and then substitute (x^*,α^*), we get

$$\frac{\partial B}{\partial x}(x^*,\alpha^*) + \frac{\partial B}{\partial \alpha}(x^*,\alpha^*)p'(x^*) = 0.$$

Since

$$\frac{\partial B}{\partial x}(x^*,\alpha^*) = \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(x^*,\alpha^*) = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x^*,\alpha^*) = 0$$

and

$$\frac{\partial B}{\partial \alpha}(x^*,\alpha^*) = \frac{\partial}{\partial \alpha}[\frac{\partial G}{\partial x}(x^*,\alpha^*)] = \frac{\partial^2 f}{\partial \alpha \partial x}(x^*,\alpha^*) \neq 0,$$

$$p'(x^*) = 0$$

So x^* is a critical point of the map $\alpha = p(x)$. Differentiate (2.1.6) again with respect to x, we get

$$\frac{\partial^2 B}{\partial x^2}(x,\alpha) + 2\frac{\partial^2 B}{\partial \alpha \partial x}(x,\alpha)p'(x) + \frac{\partial^2 B}{\partial \alpha^2}(x,\alpha)(\acute{p}(x))^2 + \frac{\partial B}{\partial \alpha}(x,\alpha)p''(x) = 0$$

Substitute (x^*,α^*) and $p'(x^*) = 0$, we have

$$\frac{\partial^2 B}{\partial x^2}(x^*,\alpha^*) + \frac{\partial B}{\partial \alpha}p''(x^*) = 0.$$

Since

$$\frac{\partial^2 B}{\partial x^2}(x^*, \alpha^*) = \frac{1}{3}\frac{\partial^3 G}{\partial x^3}(x^*, \alpha^*) = \frac{1}{3}\frac{\partial^3 f}{\partial x^3}(x^*, \alpha^*) \neq 0$$

and

$$\frac{\partial B}{\partial \alpha}(x^*, \alpha^*) = \frac{\partial}{\partial \alpha}\frac{\partial G}{\partial x}(x^*, \alpha^*) = \frac{\partial^2 f}{\partial \alpha \partial x}(x^*, \alpha^*) \neq 0,$$

$$p''(x^*) = -\frac{E}{D} \neq 0 \tag{2.1.7}$$

Formula (2.1.7) implies that if $ED < 0$, then $p'(x^*) > 0$ and the curve $p(x)$ is concave upward at $x = x^*$ and if $ED > 0$, then the curve $p(x)$ is concave downward at $x = x^*$.

Example 2.3. *Consider the map*

$$f(x, \alpha) = \alpha x - 2x^3, x \in \mathbb{R}, \alpha \in \mathbb{R}$$

.

The fixed point of this map is given by

$$h(x, \alpha) = f(x, \alpha) - x = 0$$

or

$$2x^3 - \alpha x + x = 0.$$

We have two curves of fixed points $x = 0$ *and* $x^2 = \frac{\alpha-1}{2}$. *Note that* $(0, 1)$ *is non-hyperbolic fixed point of* $f(x, \alpha)$ *such that*

$$\frac{\partial f}{\partial x}(0, 1) = 1,$$

$$\frac{\partial f}{\partial \alpha}(0, 1) = 0,$$

$$\frac{\partial^2 f}{\partial x^2}(0, 1) = 0,$$

$$\frac{\partial^2 f}{\partial x \partial \alpha}(0, 1) = 1$$

and

$$\frac{\partial^3 f}{\partial x^3}(0, 1) = -12.$$

So at $(0,1)$ *pitchfork bifurcation is present. Now we study the behavior of the system near the bifurcation point* $(0,1)$. $\mid \acute{f}(0,\alpha) \mid< 1$ *if* $-1 < \alpha < 1$ *and* $\mid \acute{f}(\pm\sqrt{\frac{\alpha-1}{2}}, \alpha) \mid< 1$ *if* $1 < \alpha < 2$. *So for* $-1 < \alpha < 1$ *we have one branch of stable fixed points* $x = 0$. *Beyond* $\alpha = 1$ *this fixed point losses its stability and two stable branches* $x = \pm\sqrt{\frac{\alpha-1}{2}}$ *appear. Beyond* $\alpha = 2$ *these two branches loses there stability.*

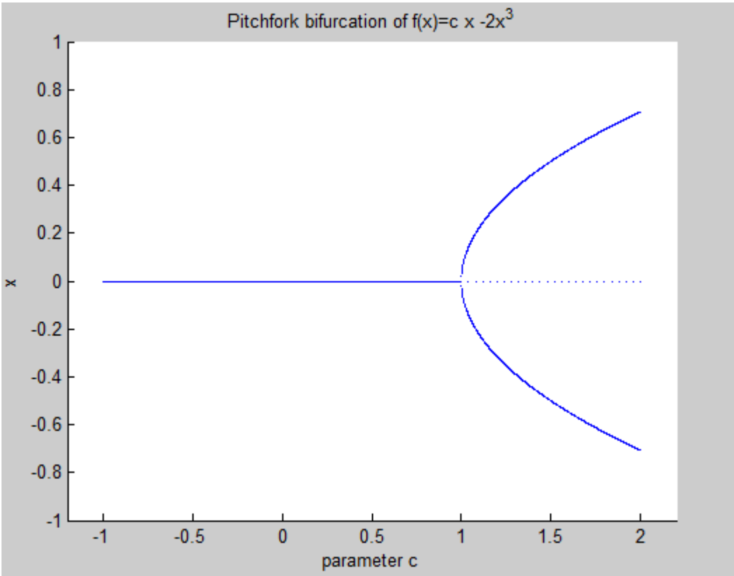

Fig. 2.3: Pitchfork bifurcation of $f(x) = cx - x^3$

2.1.4 Period-Doubling Bifurcation

Consider the one-dimensional map

$$x \to f(x, \alpha), \quad x \in \mathbb{R}, \alpha \in \mathbb{R}$$

with x^* as a non hyperbolic fixed point of $f(x, \alpha^*)$.
Period-doubling bifurcation is a type of bifurcation for the one-dimensional map f that has a nonhyperbolic fixed point (x^*, α^*) with a slope of -1 and for the second

iterate of the map f undergoes a pitchfork bifurcation at the same nonhyperbolic fixed point.

Theorem 2.5. ***[Period-Doubling Bifurcation]***[4] *Suppose that $f(x,\alpha)$ is a $C^r, r \geq 3$ one-parameter family of one-dimensional maps and x^* is a fixed point of $f(x,\alpha)$ with*

$$\frac{\partial f}{\partial x}(x^*,\alpha^*) = -1. \tag{2.1.8}$$

Assume further that

$$\frac{\partial^2 f^2}{\partial \alpha \partial x}(x^*,\alpha^*) \neq 0. \tag{2.1.9}$$

Then there is an interval J about x^* and a function $p : J \to \mathbb{R}$ such that $f(x,p(x)) \neq x$ but $f^2(x,p(x)) = x$.

Proof: Let the function $f(x,\alpha)$ be a C^2 function where $x \in \mathbb{R}, \alpha \in \mathbb{R}$ that satisfied (2.1.8) and (2.1.9).
Let

$$G(x,\alpha) = f^2(x,\alpha) - x$$

where $f^2(x,\alpha) = f(f(x,\alpha),\alpha)$. Note that

$$\frac{\partial G}{\partial \alpha}(x^*,\alpha^*) = \frac{\partial f}{\partial x}(x^*,\alpha^*)\frac{\partial f}{\partial \alpha}(x^*,\alpha^*) + \frac{\partial f}{\partial \alpha}(x^*,\alpha^*).$$

Assumption (2.1.8) implies $\frac{\partial G}{\partial \alpha}(x^*,\alpha^*) = 0$. Hence, we can not apply the Implicit Function Theorem directly to $G(x,\alpha)$. Define the function $B(x,\alpha)$ as

$$B(x,\alpha) = \begin{cases} \frac{G(x,\alpha)}{x - x^*}, & x \neq x^*; \\ \frac{\partial G}{\partial x}(x^*,\alpha), & x = x^*. \end{cases}$$

Note that

$$B(x^*,\alpha^*) = \frac{\partial G}{\partial x}(x^*,\alpha^*) = [\frac{\partial f}{\partial x}(x^*,\alpha^*)]^2 - 1.$$

From assumption (2.1.8), we get

$$B(x^*,\alpha^*) = 0.$$

$$\frac{\partial B}{\partial \alpha}(x^*,\alpha^*) = \frac{\partial}{\partial \alpha}[\frac{\partial G}{\partial x}(x^*,\alpha)] \mid_{\alpha=\alpha^*} = \frac{\partial}{\partial \alpha}[\frac{\partial f^2}{\partial x}(x^*,\alpha) - 1] \mid_{\alpha=\alpha^*} = \frac{\partial^2 f^2}{\partial \alpha \partial x}(x^*,\alpha^*).$$

Assumption (2.1.9) implies

$$\frac{\partial B}{\partial \alpha}(x^*, \alpha^*) \neq 0.$$

Now, we can apply the Implicit Function Theorem to $B(x, \alpha)$. By the Implicit Function Theorem, there is an interval J around x^* and a C^1 map $p : J \to \mathbb{R}$ such that $p(x^*) = \alpha^*$ and

$$B(x, p(x)) = 0, \quad \forall x \in J$$

and so

$$\frac{G(x, p(x))}{x - x^*} = 0$$
$$f^2(x, p(x)) = x,$$

so x is a two-periodic point of $f(x, \alpha)$.

Example 2.4. *Consider the map*

$$f(x, \alpha) = 2x^3 + x - \alpha x, \quad x \in \mathbb{R}^1, \alpha \in \mathbb{R}^1.$$

Fixed points of $f(x, \alpha)$ are the roots of the function

$$h(x, \alpha) = f(x, \alpha) - x = 2x^3 + x - (\alpha + 1)x.$$

Hence, $f(x, \alpha)$ has two curves of fixed points $x = 0$ and $x^2 = \frac{\alpha}{2}$. The curve $x = 0$ is stable if $0 < \alpha < 2$ and the curve $x^2 = \frac{\alpha}{2}$ does not exist if $\alpha < 0$ and unstable for $\alpha \geq 0$.
Note that $(0, 2)$ is a non-hyperbolic fixed point since

$$f(0, 2) = 0 \quad \textit{and} \quad \frac{\partial f}{\partial x}(0, 2) = -1$$

Thus for $\alpha > 2$, the map has three unstable curves of fixed points. Period-doubling bifurcation may be present at $(0, 2)$. The two-periodic points are the roots of the function

$$g(x, \alpha) = f^2(x, \alpha) - x$$

where

$$f^2(x, \alpha) = (\alpha - 1)^2 x - 2(\alpha^3 - 3\alpha^2 + 4\alpha - 2)x^3 + \mathcal{O}(4)$$

Observe that

$$f^2(0,2) = 0,$$
$$\frac{\partial f^2}{\partial x}(0,2) = 1,$$
$$\frac{\partial f^2}{\partial \alpha}(0,2) = 0,$$
$$\frac{\partial^2 f^2}{\partial x^2}(0,2) = 0,$$
$$\frac{\partial^3 f^2}{\partial x^3}(0,2) \neq 0$$

and

$$\frac{\partial^2 f^2}{\partial x \partial \alpha}(0,2) \neq 0.$$

Thus $(0,2)$ *is a non-hyperbolic fixed point of the map* $f^2(x,\alpha)$ *where this map undergoes a pitchfork bifurcation. So period-doubling bifurcation is present at* $(0,2)$.

We can use the normal form of flip bifurcation theorem to check if the system undergoes a period-doubling (flip) bifurcation. We will study the normal form theorem for flip bifurcation in the simplest form.

The normal form of the period-doubling (flip) bifurcation

Consider the following one-dimensional dynamical system depending on one parameter :

$$x \mapsto -(1+\alpha)x + x^3 \equiv f(x,\alpha).$$

The map $f(x,\alpha)$ is invertible for small $\mid \alpha \mid$ in a neighborhood of the origin. $f(x,\alpha)$ has the fixed point $x_0 = 0$ for all α with eigenvalue $\mu = -(1+\alpha)$. The point is linearly stable for small $\alpha < 0$ and is linearly unstable for $\alpha > 0$. Note that at $\alpha = 0$ $\mu = f_x(0,0) = -1$ so the point is nonhyperbolic. There are no other fixed point

near the origin for small $\mid \alpha \mid$.
Consider the second iterate $f^2(x,\alpha)$. If $y = f(x,\alpha)$, then

$$f^2(x,\alpha) = f(y,\alpha) = -(1+\alpha)y + y^3$$

$$= -(1+\alpha)[-(1+\alpha)x + x^3] + [-(1+\alpha)x + x^3]^3$$

$$= (1+\alpha)^2 x - [(1+\alpha)(2+2\alpha+\alpha^2)]x^3 + O(x^5).$$

Note that the map $f^2(x,\alpha)$ has the trivial fixed point x_0. It also has two nontrivial fixed points for small $\alpha > 0$

$$x_{1,2} = f^2(x_{1,2},\alpha)$$

where $x_{1,2} = \pm(\sqrt{\alpha} + O(\alpha))$. These two points are stable and constitute a cycle of period two for the original map $f(x,\alpha)$ such that

$$x_1 = f(x_2,\alpha), \quad x_2 = f(x_1,\alpha).$$

As α approaches zero from above, the period two cycle shrinks and disappears. This is a period-doubling (flip) bifurcation and it is called in this case supercritical. Note that trivial fixed point is stable for $\alpha < 0$ and the period-two cycle x_1, x_2 existing for $\alpha > 0$.
The case

$$x \mapsto -(1+\alpha)x - x^3$$

can be treated in the same way and the flip bifurcation is called subcritical in this case.[1]

Generic flip bifurcation

Theorem 2.6. *[1] Suppose that a one-dimensional map*

$$x \mapsto f(x,\alpha), \quad x \in \mathbb{R}, \alpha \in \mathbb{R}$$

with smooth f, has at $\alpha = 0$ the fixed point $x_0 = 0$ and let $\mu = f_x(0,0) = -1$. Assume that the following nondegeneracy conditions hold:

1. $\frac{1}{2}(f_{xx}(0,0))^2 + \frac{1}{3}f_{xxx}(0,0) \neq 0,$

2. $f_{x\alpha}(0,0) \neq 0.$

Then there are smooth invertible coordinate and parameter changes transforming the system into

$$\eta \mapsto -(1+\beta)\eta \pm \eta^3 + O(\eta^4).$$

Proof: By the Implicit Function Theorem, the system has a unique fixed point $x_0(\alpha)$ in some neighborhood of the origin for all sufficiently small $\mid \alpha \mid$, since $f_x(0,0) \neq 1$. We can perform a coordinate shift, placing this fixed point at the origin. Therefore, we can assume without loss of generality that $x = 0$ is the fixed point of the system for $\mid \alpha \mid$ sufficiently small. Thus, the map f can be written as follows:

$$f(x,\alpha) = f_x(0,\alpha)x + \frac{1}{2}f_{xx}(0,\alpha)x^2 + \frac{1}{6}f_{xxx}(0,\alpha)x^3 + O(x^4), \tag{2.1.10}$$

where $f_x(0,\alpha) = -[1 + g(\alpha)]$ for some smooth function g. Since $g(0) = 0$ and

$$\acute{g}(0) = f_{x\alpha}(0,0) \neq 0,$$

the function g is locally invertible and can be used to introduce a new parameter:

$$\beta = g(\alpha).$$

Map (2.1.10) can be written as

$$\tilde{x} = \mu(\beta)x + a(\beta)x^2 + b(\beta)x^3 + O(x^4),$$

where $\mu(\beta) = -(1+\beta)$, and the functions $a(\beta)$ and $b(\beta)$ are smooth and equal

$$a(0) = \frac{1}{2}f_{xx}(0,0), \quad b(0) = \frac{1}{6}f_{xxx}(0,0).$$

Define a smooth function $\delta = \delta(\beta)$ and make a change of coordinate

$$x = y + \delta y^2.$$

This transformation is invertible in some neighborhood of the origin and it's inverse can be found by the method of unknown coefficients:

$$y = x - \delta x^2 + 2\delta^2 x^3 + O(x^4).$$

Using the previous transformation and it's inverse, we obtain

$$\tilde{y}\mu y + (a + \delta\mu - \delta\mu^2)y^2 + (b + 2\delta a - 2\delta\mu(\delta\mu + a) + 2\delta^2\mu^3)y^3 + O(y^4).$$

Setting

$$\delta(\beta) = \frac{a(\beta)}{\mu^2(\beta) - \mu(\beta)}.$$

Since $\mu^2(0) - \mu(0) = 2 \neq 0$, the quadratic term is killed for all sufficiently small $\mid \beta \mid$. We have

$$\tilde{y} = \mu y + (b + \frac{2a^2}{\mu^2 - \mu})y^3 + O(y^4) = -(1+\beta)y + c(\beta)y^3 + O(y^4)$$

where $c(\beta)$ is a smooth function such that

$$c(0) = a^2(0) + b(0) = \frac{1}{4}(f_{xx}(0,0))^2 + \frac{1}{6}f_{xxx}(0,0).$$

Since we assume $\frac{1}{2}(f_{xx}(0,0))^2 + \frac{1}{3}f_{xxx}(0,0) \neq 0$, $c(0) \neq 0$.
Take

$$y = \frac{\eta}{\sqrt{\mid c(\beta) \mid}}.$$

The map takes the desired form

$$\tilde{\eta} = -(1+\beta)\eta + s\eta^3 + O(\eta^4).$$

where $s = \text{sign } c(0) = \pm 1.\diamond$

Lemma 2.7. *[1] The map*

$$x \mapsto -(1+\beta)x + x^3 + O(x^4)$$

is locally topologically equivalent near the origin to the map

$$x \mapsto -(1+\beta)x + x^3. \quad \diamond$$

We have the following general result.

Theorem 2.8. *[1](**Topological normal form for the flip bifurcation**)*
Any generic one-parameter map

$$x \mapsto f(x, \alpha)$$

having at $\alpha = 0$ *the fixed point* $x_0 = 0$ *with* $\mu = f_x(0,0) = -1$*, is locally topologically equivalent near the origin to one of the following normal forms:*

$$\eta \mapsto -(1+\beta)\eta \pm \eta^3. \quad \diamond$$

In the flip case for any n-dimensional map

$$x = Ax + F(x), \quad x \in \mathbb{R}^n \tag{2.1.11}$$

where $F(x) = O(\| x \|^2)$ is a smooth function and and its Taylor expansion is

$$F(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O(\| x \|^4)$$

where

$$B_i(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 X_i(\xi)}{\partial \xi_j \partial \xi_k} |_{\xi=0} (x_j y_k) \quad and \quad C_i(x,y,z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 X_i(\xi)}{\partial \xi_j \partial \xi_k} |_{\xi=0} (x_j y_k z_l).$$

, and the Jacobian matrix A has the eigenvalue $\mu = -1$ and the corresponding eigenspace T^c is one-dimensional and spanned by an eigenvector $q \in \mathbb{R}^n$ such that $Aq = \mu q$. Let $p \in \mathbb{R}^n$ be the adjoint eigenvector, such that $A^T p = \mu p$ where A^T is the transposed matrix. Normalize p with respect to q such that $< p, q >= 1$. Let T^{su} denote an $(n-1)-$ dimensional linear eigenspace of A corresponding to all eigenvalues other than μ. Note that the matrix $A - \mu I_n$ has common invariant spaces with the matrix A, we conclude that $y \in T^{su}$ if and only if $< p, y >= 0$.
Any vector $x \in \mathbb{R}^n$ can be decomposed as

$$x = uq + y$$

where $uq \in T^c, y \in T^{su}$ and

$$u =< p, x >$$

$$y = x - < p, x > q. \tag{2.1.12}$$

In the coordinates (u, y), the map (2.1.11) can be written as

$$\tilde{u} = \mu u + < p, F(uq + y) >,$$

$$\tilde{y} = Ay + F(uq + y) - < p, F(uq + y) > q. \tag{2.1.13}$$

Using Taylor expansions, (2.1.13) can be written as

$$\tilde{u} = \mu u + \frac{1}{2}\sigma u^2 + u < b, y > + \frac{1}{6}\delta u^3 + \ldots$$

$$\tilde{y} = Ay + \frac{1}{2}au^2 + \ldots \tag{2.1.14}$$

where $u \in \mathbb{R}^1, y \in \mathbb{R}^n, \sigma, \delta \in \mathbb{R}^1, a, b \in \mathbb{R}^n$ and $< b, y >= \sum_{i=1}^{n} b_i y_i$ is the standard scaler product $< b, y >$ can be expressed as

$$< b, y >=< p, B(q, y) > .$$

The center manifold of (2.1.14) has the representation

$$y = V(u) = \frac{1}{2}w_2 u^2 + O(u^3),$$

where $w_2 \in T^{su} \subset \mathbb{R}^n$, so that $< p, w >= 0$. The vector w_2 satisfies

$$(A - I_n)w_2 + a = 0.$$

Note that the matrix $A - I_n$ is invertible in $\mathbb{R}^n$ because $\mu = 1$ is not an eigenvalue of A. Thus, we have

$$w_2 = -(A - I_n)^{-1}a$$

and the restriction of (2.1.14) to the center manifold takes the form

$$\tilde{u} = -u + \frac{1}{2}\sigma u^2 + \frac{1}{6}(\delta - 3 < p, B(q, (A - I)^{-1}a) >)u^3 + O(u^4)$$

where $\sigma =< p, B(q, q) >, \delta =< p, C(q, q, q) >$ and $a = B(q, q) - < p, B(q, q) > q$.

Using the identity $(A - I_n)^{-1}q = -\frac{1}{2}q$, the restricted map can be written as

$$\tilde{u} = -u + a(0)u^2 + b(0)u^3 + O(u^4), \tag{2.1.15}$$

where

$$a(0) = \frac{1}{2} < p, B(q,q) >$$

and

$$b(0) = \frac{1}{6} < p, C(q,q,q) > -\frac{1}{4}(< p, B(q,q) >)^2 - \frac{1}{2} < p, B(q,(A - I_n)^{-1}B(q,q)) > .$$

The map (2.1.15) can be transformed to the normal form

$$\tilde{\xi} = -\xi + c(0)\xi^3 + O(\xi^4)$$

where

$$c(0) = a^2(0) + b(0).$$

Thus, the critical normal form coefficient $c(0)$ allows us to predict the direction of bifurcation of period-two cycle. $c(0)$ is given by the following invariant formula:

$$c(0) = \frac{1}{6} < p, C(q,q,q) > -\frac{1}{2} < p, B(q,(A - I)^{-1}B(q,q)) > .$$

If $c(0) > 0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point.[1]

Note that in example (2.4), theorem (2.6) implies that period-doubling bifurcation exists, since $\frac{1}{2}(f_{xx}(0,2))^2 + \frac{1}{3}f_{xxx}(0,2) = \frac{1}{3}(12) = 4 \neq 0$ and $f_{x\alpha}(0,2) = -1 \neq 0$.

2.2 Bifurcation Of Two-Dimensional Maps

Two-dimensional maps have the same types of bifurcation of one-dimensional maps in addition to a new type which has no analogue in the one-dimensional maps. It is the Neimark-Saker bifurcation. In this section we will discuss in details Neimark-Sacker bifurcation in the simplest form.

Non-hyperbolic fixed points of the two-dimensional maps are those where the Jacobian matrix has eigenvalue on the unit circle.

Let

$$f(x,\alpha), x \in \mathbb{R}^2, \alpha \in \mathbb{R}$$

be a $C^r, r \geq 5$ one-parameter family of two-dimensional maps and let (x^*, α^*) be a fixed point of $f(x, \alpha)$. If $(x^*, \alpha^*) \neq (0, 0)$, we transform this fixed point to the origin. Let $A = Jf(0, 0)$ be the Jacobian matrix of $f(x, \alpha)$ and let $\rho(A) = 1$. There are three cases to consider

1. if $Jf(0, 0)$ has one real eigenvalue equals to 1, then we have one of the following bifurcation (saddle-node, transcritical or pitchfork bifurcation).
2. if $Jf(0, 0)$ has one real eigenvalue equal to -1, then we have a period-doubling bifurcation.
3. if $Jf(0, 0)$ has two complex conjugate eigenvalues with modulus equal to 1, then Neimark-Sacker bifurcation appears.[4]

We show in the previous chapter that any fixed point of two-dimensional system is stable in the region enclosed by the lines: $\det A = -trA - 1$, $\det A = trA - 1$ and $\det A = 1$ where A is the Jacobian matrix evaluated at this fixed point. Thus when trA and $\det A$ pass through those lines toward the region of stability, the fixed point becomes stable.

Theorem 2.9. *[4] Consider the map*

$$x \to f(x, \alpha), x \in \mathbb{R}^2, \alpha \in \mathbb{R} \tag{2.2.1}$$

Let $A = Jf(x^*, \alpha^*)$ where (x^*, α^*) is a fixed point of $f(x, \alpha)$. Then the following hold

1. *If $\det A = -trA - 1$, then the eigenvalues of A are $\lambda_1 = -\det A$ and $\lambda_2 = -1$.*
2. *If $\det A = trA - 1$, then $\lambda_1 = 1$ and $\lambda_2 = \det A$.*
3. *If $\mid trA \mid -1 < \det A$ and $\det A = 1$, then A has complex eigenvalues $\lambda_{1,2} = e^{\pm i\theta}$ where $\theta = cos^{-1}(\frac{trA}{2})$.*

Proof: Consider the map (2.2.1) with the Jacobian matrix $A = Jf(x^*, \alpha^*)$. Recall that the eigenvalues of A are

$$\lambda_{1,2} = \frac{1}{2}[trA \pm \sqrt{(trA)^2 - 4detA}]$$

1. Let $\det A = -trA - 1$. Then $(trA)^2 - 4\det A = (\det A + 1)^2 - 4\det A = (\det A - 1)^2$. This term is non-negative. This implies that A has real eigenvalues and hence

   $$\lambda_{1,2} = \frac{1}{2}[-1 - \det A \pm \sqrt{(\det A - 1)^2}] = \frac{1}{2}[-1 - \det A \pm (\det A - 1)].$$

   So

   $$\lambda_1 = -1, \lambda_2 = -detA$$

2. Let $\det A = trA - 1$. Then $(trA)^2 - 4detA = (\det A - 1)^2 > 0$. Hence

   $$\lambda_{1,2} = \frac{1}{2}[\det A + 1 \pm \sqrt{(\det A - 1)^2}] = \frac{1}{2}[\det A + 1 \pm (\det A - 1)].$$

   This implies that

   $$\lambda_1 = \det A, \lambda_2 = 1$$

3. Let $\mid trA \mid -1 < \det A$ and $\det A = 1$. Then

   $$(trA)^2 - 4\det(A) = (trA)^2 - 4 < (\det A + 1)^2 - 4 = 0.$$

   This implies that A has two complex conjugate eigenvalues

   $$\lambda_{1,2} = \frac{1}{2}[trA \pm \sqrt{4\det A - (trA)^2}].$$

   Substitute $\det A = 1$, we have

   $$\lambda_{1,2} = \frac{1}{2}[trA \pm \sqrt{4 - (trA)^2}].$$

   Therefore $\lambda_{1,2} = re^{\pm i\theta}$ where $r =\mid \lambda_{1,2} \mid = \sqrt{(\frac{trA}{2})^2 + \frac{4-(trA)^2}{4}} = 1$ and $\theta = tan^{-1}(\frac{\pm\frac{\sqrt{4-(trA)^2}}{4}}{\frac{(trA)^2}{4}}) = cos^{-1}(\frac{trA}{2})$. $\diamond$

Corollary 2.9.1. *For the one-parameter of two-dimensional map*

$$x \to f(x, \alpha), x \in \mathbb{R}^2, \alpha \in \mathbb{R} \tag{2.2.2}$$

with the fixed point (x^*, α^*) and $A = Jf(x^*, \alpha^*)$, then the following hold

1. *If $\det A = -trA - 1$, then the system (2.2.2) undergoes a period-doubling bifurcation.*
2. *If $\det A = trA - 1$, then then the system (2.2.2) undergoes a saddle-node bifurcation.*
3. *If $\mid trA \mid -1 < \det A$ and $\det A = 1$, then the system (2.2.2) undergoes a Neimark-Sacker bifurcation.*

This corollary will help us in studying the bifurcation of the second-order difference equation $x_{n+1} = \frac{\alpha+\beta x_{n-1}}{A+Bx_n+Cx_{n-1}}$.

2.2.1 A Pair Of Eigenvalue Of Modulus 1:The Neimark-sacker bifurcation

We turn our attention to the Neimark-Sacker bifurcation which exists in the case that we have a complex-conjugate pair of eigenvalues of modulus equals 1.
Any map undergoes the Neimark-Sacker bifurcation has a unique closed invariant curve bifurcates from the fixed point as the bifurcation parameter passes through zero. The closed invariant curve can be stable or unstable as the bifurcation is supercritical or subcritical, respectively.

2.2.2 The normal form of Neimark-Sacker bifurcation

Consider the two dimensional discrete-time system depending on one parameter

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to (1+\alpha)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+(x_1^2+x_2^2)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.2.3}$$

where α is the parameter, $\theta = \theta(\alpha)$, $a = a(\alpha)$ and $b = b(\alpha)$ are smooth functions and $0 < \theta(0) < \pi$, $a(0) \neq 0$.
This system has the fixed point $x_1 = x_2 = 0$ for all α with Jacobian matrix

$$A = (1+\alpha)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The Jacobian matrix A has the eigenvalues $\mu_{1,2} = (1+\alpha)e^{\pm i\theta}$ which makes the map (2.2.3) is invertible near the origin for all small α. Note that at $\alpha = 0$, A has a complex-conjugate pair of eigenvalues of modulus one. So at $\alpha = 0$ the origin is non-hyperbolic.
To analyze the corresponding bifurcation, introduce the complex variable $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ and let $d = a + ib$. Note that $\mid z \mid^2 = z\bar{z} = x_1^2 + x_2^2$. The equation for z is

$$z \to e^{i\theta} z(1 + \alpha + d \mid z \mid^2) = \mu z + cz \mid z \mid^2$$

where $\mu = \mu(\alpha) = (1+\alpha)e^{i\theta(\alpha)}$ and $c = c(\alpha) = e^{i\theta(\alpha)}d(\alpha)$ are complex functions of the parameter α. Using the representation $z = \rho e^{i\varphi}$ where $\rho = \mid z \mid$. We obtain

$$\rho \to \rho \mid 1 + \alpha + d(\alpha)\rho^2 \mid$$

Note that $\mid 1+\alpha+d(\alpha)\rho^2 \mid = (1+\alpha)\sqrt{1 + \frac{2a(\alpha)}{1+\alpha}\rho^2 + \frac{|d(\alpha)|^2}{(1+\alpha)^2}\rho^4} = 1+\alpha+a(\alpha)\rho^2+O(\rho^3)$. We obtain the following polar form

$$\rho \to \rho(1 + \alpha + a(\alpha)\rho^2) + \rho^4 R_\alpha(\rho)$$

$$\varphi \to \varphi + \theta(\alpha) + \rho^2 Q_\alpha(\rho).$$

Where R and Q are smooth functions of (ρ, α). Since the mapping for ρ is independent of φ, bifurcations of the system's phase portraits as α passes through zero can easily be analyzed using the latter form. Consider the first equation

$$\rho \to \rho(1 + \alpha + a(\alpha)\rho^2 + \rho^4 R_\alpha(\rho)).$$

It defines a one-dimensional dynamical system with $\rho = 0$ as a fixed point for all values of α. The fixed point losses it's stability as α becomes positive. The stability of the fixed point at $\alpha = 0$ is determined by the sign of the $a(0)$. If $a(0) < 0$, then the origin is stable at $\alpha = 0$. In this case, the previous $\rho-$map has an additional stable fixed point

$$\rho_0(\alpha) = \sqrt{-\frac{\alpha}{a(\alpha)}} + O(\alpha)$$

for $\alpha > 0$. Consider the second equation

$$\varphi \to \varphi + \theta(\alpha) + \rho^2 Q_\alpha(\rho)$$

which describes a rotation by an angle depending on ρ and α; it is approximately equal to $\theta(\alpha)$. Thus, we obtain the bifurcation diagram for the two dimensional system (2.2.3) by superposition of it's polar form.

Also system (2.2.3) has a fixed point at the origin which is stable for $\alpha < 0$ and unstable for $\alpha > 0$. The invariant curve of the system near the origin look like the orbits near the stable focus of a continuous-time system for negative values of α and like orbits near the unstable focus for positive values of α. At $\alpha = 0$ the point is nonlinearly stable. The fixed point is surrounded for $\alpha > 0$ by an isolated, unique and stable closed invariant curve which is a circle of radius $\rho_0(\alpha)$. All orbits is starting outside or inside the closed invariant curve, except at the origin, tend to the curve under iterations of (2.2.3). This bifurcation can be presented in (x_1, x_2, α)-space. The appearing family of closed invariant curves, parameterized by α, forms a paraboloid surface. This is Neimark-Sacker bifurcation.

If $a(\alpha) > 0$ can be analyzed in the same way. The system (2.2.3) undergoes the Neimark-Sacker bifurcation at $\alpha = 0$ but there is an unstable closed invariant curve that disappears when α crosses zero from negative to positive values.[1]

2.2.3 *Generic Neimark-Sacker bifurcation*

We now shall prove that any generic two-dimensional system undergoing a Neimark-Sacker bifurcation can be transformed into the form (2.2.3).

Consider the system

$$x \mapsto f(x, \alpha), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \alpha \in \mathbb{R}$$

with a smooth function f, where f has the fixed point $x = 0$ at $\alpha = 0$with simple eigenvalues $\mu_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$. By the Implicit Function Theorem, the system has a unique fixed point $x_0(\alpha)$ in some neighborhood of the origin for all sufficiently small $\mid \alpha \mid$, since $\mu = 1$ is not an eigenvalue of the Jacobian matrix. We can perform a parameter-dependent coordinate shift, placing this point at the origin. Therefore, we assume without loss of generality that $x = 0$ is the fixed point of the system for $\mid \alpha \mid$ sufficiently small. Thus the system can be written as

$$x \mapsto A(\alpha)x + F(x, \alpha) \tag{2.2.4}$$

where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in x starting with at least quadratic terms, $F(0, \alpha) = 0$ for all sufficiently small $\mid \alpha \mid$. The Jacobian matrix $A(\alpha)$ has two multipliers

$$\mu_{1,2} = r(\alpha)e^{\pm i\varphi(\alpha)}$$

where $r(0) = 1$, $\varphi(0) = \theta_0$. Thus, $r(\alpha) = 1 + \beta(\alpha)$ for some smooth function $\beta(\alpha)$ such that $\beta(0) = 0$. Suppose that $\acute{\beta}(0) \neq 0$. Then, we can use β as a new parameter and express the multipliers in terms of $\beta : \mu_1(\beta) = \mu(\beta), \mu_2(\beta) = \bar{\mu}(\beta)$, where

$$\mu(\beta) = (1 + \beta)e^{i\theta(\beta)}$$

with a smooth function $\theta(\beta)$ such that $\theta(0) = \theta_0$.

Lemma 2.10. *[1] By the introduction of a complex variable and a new parameter, system (2.2.4) can be transformed for all sufficiently small* $\mid \alpha \mid$ *into the form*

$$z \mapsto \mu(\beta)z + g(z, \bar{z}, \beta), \tag{2.2.5}$$

where $\beta \in \mathbb{R}^1, z \in \mathbb{C}^1, \mu(\beta) = (1+\beta)e^{i\theta(\beta)}$, *and* g *is a complex-valued smooth function of* $z, \bar{z}$, *and* β *whose Taylor expansion with respect to* $(z, \bar{z})$ *contains quadratic and higher-order terms:*

$$g(z, \bar{z}, \beta) = \sum_{k+l\geq 2} \frac{1}{k!l!} g_{kl} z^k \bar{z}^l,$$

with $l, k = 0, 1, 2, \ldots$*.[1]*

Lemma 2.11. *[1] The map*

$$z \mapsto \mu z + \frac{g_{20}}{2} z^2 + g_{11} z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + O(\mid z \mid^3) \tag{2.2.6}$$

where $\mu = \mu(\beta) = (1+\beta)e^{i\theta(\beta)}, g_{ij} = g_{ij}(\beta)$ *can be transformed by an invariant parameter-dependent change of complex coordinate*

$$z = w + \frac{h_{20}}{2} w^2 + h_{11} w\bar{w} + \frac{h_{02}}{2}\bar{w}^2,$$

for all sufficiently small $\mid \beta \mid$*, into a map without quadratic terms:*

$$w \mapsto \mu w + O(\mid w \mid^3),$$

provided that

$$e^{i\theta_0} \neq 1, \quad e^{3i\theta_0} \neq 1$$

Proof: The inverse change of variable is given by

$$w = z - \frac{h_{20}}{2} z^2 - h_{11} z\bar{z} - \frac{h_{02}}{2}\bar{z}^2 + O(\mid z \mid^3).$$

The new coordinate w implies that the map (2.2.6) takes the form

$$\tilde{w} = \mu w + \frac{1}{2}(g_{20} + (\mu - \mu^2)h_{20})w^2 + (g_{11} + (\mu - \mid \mu \mid^2)h_{11})w\bar{w}$$
$$+\frac{1}{2}(g_{02} + (\mu - \bar{\mu}^2)h_{02})\bar{w}^2 + O(\mid w \mid^3).$$

Thus, by setting

$$h_{20} = \frac{g_{20}}{\mu^2 - \mu}, h_{11} = \frac{g_{11}}{\mid \mu \mid^2 - \mu}, h_{02} = \frac{g_{02}}{\bar{\mu}^2 - \mu},$$

we kill all the quadratic terms in (2.2.6). These substitutions are valid if the denominators are nonzero for all sufficiently small β including $\beta = 0$. Indeed, this is the case, since

$$\mu^2(0) - \mu(0) = e^{i\theta_0}(e^{i\theta_0} - 1) \neq 0,$$

$$| \mu(0) |^2 - \mu(0) = 1 - e^{i\theta_0} \neq 0,$$

$$\bar{\mu}^2(0) - \mu(0) = e^{i\theta_0}(e^{-3i\theta_0} - 1) \neq 0$$

due to our restrictions on θ_0. $\diamond$

Assuming that all quadratic terms are removed. Now, we will try to remove the cubic terms.

Lemma 2.12. *[1] The map*

$$z \mapsto \mu z + \frac{g_{30}}{6} z^3 + \frac{g_{21}}{2} z^2 \bar{z} + \frac{g_{12}}{2} z \bar{z}^2 + \frac{g_{03}}{6} \bar{z}^3 + O(| z |^4),$$

where $\mu = \mu(\beta) = (1 + \beta)e^{i\theta(\beta)}, g_{ij} = g_{ij}(\beta)$, can be transformed by an invertible parameter-dependent change of coordinates

$$z = w + \frac{h_{30}}{6} w^3 + \frac{h_{21}}{2} w^2 \bar{w} + \frac{h_{12}}{2} w \bar{w}^2 + \frac{h_{03}}{6} \bar{w}^3,$$

for all sufficiently small $| \beta |$, into a map with only one cubic term:

$$w \mapsto \mu w + c_1 w^2 \bar{w} + O(| w |^4),$$

provided that

$$e^{2i\theta_0} \neq 1, \quad e^{4i\theta_0} \neq 1.$$

Proof: The inverse transformation is

$$w = z - \frac{h_{30}}{6} z^3 - \frac{h_{21}}{2} z^2 \bar{z} - \frac{h_{12}}{2} \bar{z}^2 z - \frac{h_{03}}{6} \bar{z}^3 + O(| z |^4)$$

Therefore,

$$\tilde{w} = \mu w + \frac{1}{6}(g_{30} + (\mu - \mu^3)h_{30})w^3 + \frac{1}{2}(g_{21} + (\mu - \mu \mid \mu \mid^2)h_{21})w^2 \bar{w}$$

$$+\frac{1}{2}(g_{12} + (\mu - \bar{\mu} \mid \mu \mid^2)h_{12})w \bar{w}^2 + \frac{1}{6}(g_{03} + (\mu - \bar{\mu}^3)h_{03})\bar{w}^3 + O(| w |^4).$$

Thus, by setting

$$h_{30} = \frac{g_{30}}{\mu^3 - \mu}, h_{12} = \frac{g_{12}}{\bar{\mu} \mid \mu \mid^2 - \mu}, h_{03} = \frac{g_{03}}{\bar{\mu}^3 - \mu},$$

we can eliminate all the cubic terms in the resulting map except the $w^2\bar{w}$-term, which must be treated separately. The substitutions are valid since all the assumptions concerning θ_0. $\diamond$

One can also try to eliminate the $w^2\bar{w}$-term by formally setting

$$h_{21} = \frac{g_{21}}{\mu(1- \mid \mu \mid^2)}$$

This is possible for small $\beta \neq 0$, but the denominator vanishes at $\beta = 0$ for all θ_0. Thus, no extra conditions on θ_0 would help. To obtain a transformation that is smoothly dependent on β, set $h_{21} = 0$, that results in

$$c_1 = \frac{g_{21}}{2}. \quad \diamond$$

Lemma 2.13. *[1]* ***(Normal form for the Neimark-Sacker bifurcation)***
The map

$$z \mapsto \mu z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + \frac{g_{30}}{6}z^3 + \frac{g_{21}}{2}z^2\bar{z} + \frac{g_{12}}{2}z\bar{z}^2 + \frac{g_{03}}{6}\bar{z}^3 + O(\mid z \mid^4)$$

where $\mu = \mu(\beta) = (1+\beta)e^{i\theta(\beta)}, g_{ij} = g_{ij}(\beta)$, *and* $\theta_0 = \theta(0)$*is such that* $e^{ik\theta_0} \neq 1$ *for* $k = 1, 2, 3, 4$, *can be transformed by an invertible parameter dependent change of complex coordinate, which is smoothly dependent on the parameter,*

$$z = w + \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2 + \frac{h_{30}}{6}w^3 + \frac{h_{12}}{2}w\bar{w}^2 + \frac{h_{03}}{6}\bar{w}^3$$

for all sufficiently small $\mid \beta \mid$, *into a map with only the resonant cubic term:*

$$w \mapsto \mu w + c_1 w^2\bar{w} + O(\mid w \mid^4)$$

where $c_1 = c_1(\beta)$. $\diamond$

The truncated superposition of the transformations defined in the two previous lemmas gives the required coordinate change. First, annihilate terms. The coefficient of $w^2\bar{w}$ will be $\frac{1}{2}\tilde{g}_{21}$, say, instead of $\frac{1}{2}g_{21}$. Then, eliminate all the cubic terms except the resonant one. The coefficient of this term remains $\frac{1}{2}\tilde{g}_{21}$. Thus, all we need to compute to get the coefficient of c_1 in terms of the given equation is a new coefficient

$\frac{1}{2}\tilde{g}_{21}$ of the $w^2\bar{w}$-term after the quadratic transformation. The computations results in the expression for $c_1(\alpha)$

$$c_1 = \frac{g_{20}g_{11}(\bar{\mu}-3+2\mu)}{2(\mu^2-\mu)(\bar{\mu}-1)} + \frac{\mid g_{11}\mid^2}{1-\bar{\mu}} + \frac{\mid g_{02}\mid^2}{2(\mu^2-\bar{\mu})} + \frac{g_{21}}{2},$$

which gives, for the critical values of c_1,

$$c_1(0) = \frac{g_{20}(0)g_{11}(0)(\bar{\mu}-3+2\mu_0)}{2(\mu_0^2-\mu_0)(\bar{\mu}_0-1)} + \frac{\mid g_{11}(0)\mid^2}{1-\bar{\mu}_0} + \frac{\mid g_{02}(0)\mid^2}{2(\mu_0^2-\bar{\mu}_0)} + \frac{g_{21}(0)}{2}, \qquad (2.2.7)$$

where $\mu_0 = e^{i\theta_0}$.

We summarize the obtained results in the next theorem.

Theorem 2.14. *[1] Suppose a two-dimensional discrete-time system*

$$x \mapsto f(x,\alpha), \quad x \in \mathbb{R}^2, \alpha \in \mathbb{R}^1$$

with smooth function f which has, for all sufficiently small $\mid \alpha \mid$, $x = 0$ as a fixed point with eigenvalues

$$\mu_{1,2}(\alpha) = r(\alpha)e^{\pm i\varphi(\alpha)}$$

where $r(0) = 1, \varphi(0) = \theta_0$.

Let the following conditions be satisfied:

1. *$\acute{r}(0) \neq 0$;*

2. *$e^{ik\theta_0} \neq 1$ for $k = 1, 2, 3, 4$.*

Then, there are smooth invertible coordinate and parameter changes transforming the system into

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto (1+\beta)\begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$+(y_1^2+y_2^2)\begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix}\begin{pmatrix} a(\beta) & -b(\beta) \\ b(\beta) & a(\beta) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(\parallel y \parallel^4) \quad (2.2.8)$$

with $\theta(0) = \theta_0$ and $a(0) = Re(e^{-i\theta_0}c_1(0))$, where $c_1(0)$ is given by the formula (2.2.7).

Proof: The only thing left to verify is the formula for $a(0)$. Indeed, by Lemma (5.3.1), (5.3.3) and (5.3.2), the system can be transformed to the complex *Poincaré* normal form,

$$w \mapsto \mu(\beta)w + c_1(\beta)w \mid w \mid^2 + O(\mid w \mid^4),$$

where $d(\beta) = a(\beta) + ib(\beta)$ for some real functions $a(\beta), b(\beta)$. A return to the real coordinates $(y_1, y_2), w = y_1 + iy_2$, gives system (2.2.8). Finally,

$$a(\beta) = Re(e^{-i\theta(\beta)}c_1(\beta)).$$

Thus

$$a(0) = Re(e^{-i\theta_0}c_1(0)). \quad \diamond$$

Theorem 2.15. ***(Generic Neimark-Sacker bifurcation)****[1] For any generic two-dimensional one parameter system*

$$x \mapsto f(x, \alpha)$$

having at $\alpha = 0$ the fixed point $x_0 = 0$ with complex eigenvalues $m_{1,2} = e^{\pm i\theta_0}$ there is a neighborhood of x_0 in which a unique closed invariant curve bifurcates from x_0 as α passes through zero. $\diamond$

Consider the map

$$x = Ax + F(x), \quad x \in \mathbb{R}^n \tag{2.2.9}$$

where the Jacobian matrix A has simple pairs of complex eigenvalues of modulus one, $\mu_{1,2} = e^{\pm i\theta_0}, 0 < \theta_0 < \pi$ and these are the only eigenvalues with $\mid \mu \mid = 1$ and $F(x) = O(\parallel x \parallel^2)$ is a smooth function and and its Taylor expansion is

$$F(x) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + O(\parallel x \parallel^4)$$

where

$$B_i(x, y) = \sum_{j,k=1}^{n} \frac{\partial^2 X_i(\xi)}{\partial \xi_j \partial \xi_k} \mid_{\xi=0} (x_j y_k) \quad and \quad C_i(x, y, z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 X_i(\xi)}{\partial \xi_j \partial \xi_k} \mid_{\xi=0} (x_j y_k z_l).$$

Let $q \in \mathbb{C}^n$ be a complex eigenvector corresponding to μ_1:

$$Aq = e^{i\theta_0}q, A\bar{q} = e^{-i\theta_0}\bar{q}.$$

Introduce the adjoint eigenvector $p \in \mathbb{C}^n$ satisfies

$$A^T p = e^{-i\theta_0}p, \quad A^T\bar{p} = e^{i\theta_0}\bar{p}$$

and satisfies the normalization

$$< p, q >= 1$$

where $< p, q >= \sum_{i=1}^n \bar{p}_i q_i$ is the standard product in $\mathbb{C}^n$. The critical real eigenspace T^c corresponding to $\mu_{1,2}$ is two-dimensional and is spanned by $\{Re(q), Im(q)\}$. The real eigenspace T^{su} corresponding to the real eigenvalues of A is $(n-2)$-dimensional. $y \in T^{su}$ if and only if $< p, y >= 0$. Note that $y \in \mathbb{R}^n$ is real, while $p \in \mathbb{C}^n$. Any vector $x \in \mathbb{R}^n$ can be decomposed as

$$x = zq + \bar{z}\bar{q} + y$$

where $z \in \mathbb{C}^1$, $\bar{z}\bar{q} + \bar{z}\bar{q} \in T^c$ and $y \in T^s$. The complex variable z is a coordinate on T^c. We have

$$z =< p, x >$$

$$y = x- < p, x > q- < \bar{p}, x > \bar{q}.$$

In these coordinates, the map (2.2.9) takes the form

$$\tilde{z} = e^{i\theta_0}z+ < p, F(zq + \bar{z}\bar{q} + y >$$

$$\tilde{y} = Ay + F(zq + \bar{z}\bar{q} + y)- < p, F(zq + \bar{z}\bar{q} + y) > q- < \bar{p}, F(zq + \bar{z}\bar{q} + y) > \bar{q}.$$

The previous system can be written as

$$\tilde{z} = e^{i\theta_0}z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z}+ < G_{10}, y > z+ < G_{01}, y > \bar{z}$$

$$\tilde{y} = Ay + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \frac{1}{2}H_{21}z^2\bar{z}$$

where $G_{20}, G_{11}, G_{02}, G_{21} \in \mathbb{C}^1$ and $G_{01}, G_{10}, H_{ij} \in \mathbb{C}^n$ and the scaler product is in $\mathbb{C}^n$.

The complex numbers and vectors can be computed by the following formulas

$$G_{20} =< p, B(q,q) >, G_{11} =< p, B(q,\bar{q}) >, G_{02} =< p, B(\bar{q},\bar{q}) >, G_{21} =< p, C(q,q,\bar{q}) >$$

and

$$H_{20} = B(q,q)- < p, B(q,q) > q- < \bar{p}, B(q,q) > \bar{q},$$

$$H_{11} = B(q,\bar{q})- < p, B(q,\bar{q}) > q- < \bar{p}, B(q,\bar{q}) > \bar{q}$$

and

$$< G_{10}, y >=< p, B(q,y) >, < G_{01}, y >=< p, B(\bar{q},y) > .$$

From the center manifold theorem , there exists a center manifold W^c which can be approximated as

$$Y = V(z,\bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2$$

where $< q, w_{ij} >= 0$. The vectors $w_{ij} \in \mathbb{C}^n$ can be found from the linear equations

$$(e^{2i\theta_0}I - A)w_{20} = H_{20}$$

$$(I - A)w_{11} = H_{11}$$

$$(e^{-2i\theta_0}I - A)w_{02} = H_{02}$$

These equations has unique solutions. Note that the matrices $(I-A)$ and $(e^{\pm 2i\theta_0}I - A)$ are invertible in $\mathbb{C}^n$ because 1 and $e^{\pm 2i\theta_0}$ are not eigenvalues of A. Recall that $e^{i\theta_0} \neq 1$. So z can be written as

$$\tilde{z} = e^{i\theta_0}\bar{z} + \tfrac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \tfrac{1}{2}G_{02}\bar{z}^2 + \tfrac{1}{2}[G_{21} + 2 < p, B(q,(I-A)^{-1}H_{11}) >$$

$$+ < p, B(\bar{q},(e^{2i\theta_0}I - A)^{-1}H_{20}) >]z^2\bar{z} + \dots$$

Taking into account the identities

$$(I-A)^{-1}q = \frac{1}{1-e^{i\theta_0}}q, \quad (e^{2i\theta_0}I - A)^{-1}q = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}q, \quad (I-A)^{-1}\bar{q} = \frac{1}{1-e^{i\theta_0}}\bar{q}$$

and

$$(e^{2i\theta_0}I - A)^{-1}\bar{q} = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}\bar{q}.$$

Also z can be written using the map

$$\tilde{z} = e^{i\theta_0} z + \sum_{k,l\geq 2} \frac{1}{k!j!} g_{kj} z^k \bar{z}^j \tag{2.2.10}$$

where $g_{20} =< p, B(q,q) >,\ g_{11} =< p, B(q,\bar{q}) >,\ g_{02} =< p, B(\bar{q},\bar{q}) >$
and

$$g_{21} =< p, C(q,q,\bar{q}) > +2 < p, B(q,(I-J)^{-1}B(q,\bar{q})) >$$

$+ < p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q)) > +\frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{i\theta_0}} < p, B(q,q) >< p, B(q,\bar{q}) >$
$-\frac{2}{1-e^{-i\theta_0}} \mid< p, B(q,\bar{q}) >\mid^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0}-1} \mid< p, B(\bar{q},\bar{q}) >\mid^2 .$
As $e^{ik\theta_0} \neq 1$, the map (2.2.10) can be transformed into the form

$$\tilde{z} = e^{i\theta_0} z(1+d(0)) \mid z^4 \mid$$

Where $a(0) = Re\{d(0)\}$, that determines the direction of the bifurcation of the closed invariant curve, can be computed by the following formula

$$a(0) = Re(\frac{e^{-i\theta_0}g_{21}}{2}) - Re(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})} g_{20}g_{11}) - \frac{1}{2} \mid g_{11} \mid^2 - \frac{1}{4} \mid g_{02} \mid^2 .$$

This formula allows us to verify the nondegeneracy of the of the nonlinear terms at a nonresonant Neimark-Sacker bifurcation of n-dimensional maps with $n \geq 2$.[1]

Example 2.5. *Consider the map*

$$f_\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2,\alpha) \\ f_2(x_1,x_2,\alpha) \end{pmatrix} = (1+\alpha+x_1^2+x_2^2) \begin{pmatrix} cos\beta & -sin\beta \\ sin\beta & cos\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.2.11}$$

where $\beta = \beta(\alpha)$ is a smooth function of parameter α and $0 < \beta(0) < \pi$.
Note that

$$f_\alpha \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The Jacobian matrix of the system is

$$Jf = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

where

$$\frac{\partial f_1}{\partial x_1} = (1+\alpha)cos\beta + 3x_1^2 cos\beta - 2x_1x_2 sin\beta + x_2^2 cos\beta$$

$$\frac{\partial f_1}{\partial x^2} = -(1+\alpha)sin\beta - x_1^2 sin\beta + 2x_1x_2 cos\beta - 3x_2^2 sin\beta$$

$$\frac{\partial f_2}{\partial x_1} = (1+\alpha)sin\beta + 3x_1^2 sin\beta + 2x_1x_2 cos\beta + x_2^2 sin\beta$$

and

$$\frac{\partial f_2}{\partial x_2} = (1+\alpha)cos\beta + x_1^2 cos\beta + 2x_1x_2 sin\beta + 3x_2^2 cos\beta.$$

So the Jacobian matrix $Jf_\alpha(x_1,x_2)^T$ *at the fixed point* $(0,0)^T$ *is*

$$Jf_\alpha(0,0)^T = (1+\alpha)\begin{pmatrix} cos\beta & -sin\beta \\ sin\beta & cos\beta \end{pmatrix}.$$

The characteristic equation of this matrix is

$$\lambda^2 - 2\lambda cos\beta + 1 = 0$$

Hence, the eigenvalues are $\lambda_{1,2} = (1+\alpha)e^{\pm i\beta}$ *and they have a modulus equal* $\mid \lambda_{1,2} \mid = \mid 1+\alpha \mid$. *Note that at* $\alpha = 0$, *we get* $\mid \lambda_{1,2} \mid = 1$. *So at* $\alpha = 0$ *we have two complex conjugate eigenvalues of modulus one. Hence, we have a clear sign that Neimark-Sacker bifurcation maybe appear. The origin is stable if* $-2 < \alpha < 0$. *To check if the last two condition are satisfied, we write the map (2.2.11) in polar form* (r, θ). *We write equation (2.2.11) as two-dimensional system of difference equations*

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = (1+\alpha+x_1^2(n)+x_2^2(n))\begin{pmatrix} cos\beta & -sin\beta \\ sin\beta & cos\beta \end{pmatrix}\begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} \quad (2.2.12)$$

Let $x_1(n) = r(n)cos\theta(n)$, *and* $x_2(n) = r(n)sin\theta(n)$. *Substitute values of* x_1, x_2 *in equation (2.2.12), we get*

$$r(n+1) = (1+\alpha)r(n) + r^3(n)$$

$$\theta(n+1) = \theta(n) + \beta. \quad (2.2.13)$$

We will study the bifurcation of the system at $\alpha = 0$. *Note that* θ *is independent of the parameter* α *and depends on* θ *and* β. *First equation in system (2.2.13) is a one-dimensional map say* $h_\alpha(r) = (1+\alpha)r + r^3$ *which has fixed point* $r = 0$. *This fixed point is stable if* $-2 < \alpha < 0$ *and unstable for* $\alpha > 0$. *At* $\alpha = 0$,

$$h'_\alpha(0) \mid_{\alpha=0} = (1+\alpha) \mid_{\alpha=0} = 1$$

$$h''_\alpha(0) \mid_{\alpha=0} = 0$$

and

$$h'''_\alpha(0) \mid_{\alpha=0} = 6 > 0$$

so $r = 0$ *is unstable at* $\alpha = 0$. *Also the map* $h_\alpha(r)$ *has an additional fixed point* $r = \sqrt{-\alpha}$ *which is unstable closed invariant curve for* $\alpha < 0$ *which disappear when* α *vary from negative to positive value.*

3

Dynamics of $x_{n+1} = \frac{\alpha+\beta x_{n-k}}{A+Bx_n+Cx_{n-k}}$

In this chapter we will study the dynamics of the rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_{n-k}}{A + Bx_n + Cx_{n-k}}, n = 0, 1, 2, \dots \tag{3.0.1}$$

where α, β, A, B, and C are positive real numbers, the initial conditions $x_{-k}, x_{-k+1}, \dots, x_0$ are non-negative real numbers and $k \in \{1, 2, \dots\}$.
We will investigate invariant intervals, boundedness of solutions, two periodic cycles, local and global stability of positive equilibrium points.
It is worth mentioning that equation (3.0.1) for k=1 and k=2 has been investigated in [7], [2] and [6] and equation (3.0.1) has been investigated in [5].

3.1 Change of variables

Consider equation (3.0.1). Let $x_n = \frac{A}{B} y_n$, then

$$x_{n+1} = \frac{A}{B} y_{n+1}$$

and

$$x_{n-k} = \frac{A}{B} y_{n-k}.$$

Substitute in (3.0.1), we get

$$\frac{A}{B} y_{n+1} = \frac{\alpha + \beta \frac{A}{B} y_{n-k}}{A + B\frac{A}{B} y_n + C\frac{A}{B} y_{n-k}}$$

$$\frac{A}{B} y_{n+1} = \frac{(\frac{A}{B})((\frac{B}{A})\alpha + \beta y_{n-k})}{(\frac{A}{B})(B + By_n + Cy_{n-k})}$$

$$y_{n+1} = \frac{B}{A}(\frac{(\frac{B}{A})\alpha + \beta y_{n-k}}{B + By_n + Cy_{n-k}})$$

$$y_{n+1} = \frac{B}{A}(\frac{(\frac{B}{A})\alpha + \beta y_{n-k}}{B(1 + y_n + \frac{C}{B} y_{n-k})})$$

$$y_{n+1} = \frac{(\frac{B}{A^2})\alpha + \frac{\beta}{A} y_{n-k}}{1 + y_n + \frac{C}{B} y_{n-k}}$$

Let $p = \frac{B}{A^2}\alpha$, $q = \frac{B}{A}$ and $r = \frac{C}{B}$. We get

$$y_{n+1} = \frac{p + q y_{n-k}}{1 + y_n + r y_{n-k}}, \quad n = 0, 1, 2, \dots$$

3.2 Equilibrium points

In this section we find the positive equilibrium points of the rational difference equation

$$y_{n+1} = \frac{p + qy_{n-k}}{1 + y_n + ry_{n-k}}, \quad n = 0, 1, 2, \dots \tag{3.2.1}$$

where p, q and r are positive real numbers and the initial conditions $y_{-k}, y_{-k+1}, \dots, y_0$ are non-negative real numbers and $k \in \{1, 2, \dots\}$.
To find the equilibrium point of equation (3.2.1):

$$\bar{y} = \frac{p + q\bar{y}}{1 + \bar{y} + r\bar{y}}.$$

That is equivalent to

$$\bar{y}(1 + \bar{y} + r\bar{y}) = p + q\bar{y}.$$

Rearranging the terms, we get:

$$(1 + r)\bar{y}^2 + (1 - q)\bar{y} - p = 0.$$

The roots of this quadratic equation are:

$$\bar{y} = \frac{(q-1) \pm \sqrt{(q-1)^2 + 4p(1+r)}}{2(1+r)}.$$

So the positive equilibrium point is

$$\bar{y} = \frac{(q-1) + \sqrt{(q-1)^2 + 4p(1+r)}}{2(1+r)}.$$

3.3 linearized equation

Let I be any interval of real numbers and let $f : I \times I \to I$ be a continuously differentiable function. Let $\bar{y}$ be an equilibrium point of $f(x, y)$, $p = \frac{\partial f}{\partial x}(\bar{y}, \bar{y})$ and $q = \frac{\partial f}{\partial y}(\bar{y}, \bar{y})$. Then the equation

$$y_{n+1} = py_n + qy_{n-k}, n = 0, 1, 2, \dots$$

is called linearized equation associated with $y_{n+1} = f(y_n, y_{n-k}), n = 0, 1, 2, \ldots$ about the equilibrium point $\bar{y}$, and its characteristic equation is

$$\lambda^{k+1} - p\lambda^k - q = 0.$$

To find the linearized equation of (3.2.1) about the equilibrium point $\bar{y}$, let

$$f(x,y) = \frac{p+qy}{1+x+ry}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{(1+x+ry)(0)-(p+qy)}{(1+x+ry)^2}$$

$$\frac{\partial f}{\partial x}(\bar{y},\bar{y}) = \frac{-(p+q\bar{y})}{(1+\bar{y}+r\bar{y})^2}.$$

Since

$$\bar{y} = \frac{p+q\bar{y}}{1+\bar{y}+r\bar{y}},$$

we get

$$\frac{\partial f}{\partial x}(\bar{y},\bar{y}) = \frac{-\bar{y}}{1+\bar{y}+r\bar{y}}.$$

Similarly,

$$\frac{\partial f}{\partial y}(x,y) = \frac{(1+x+ry)(q)-(p+qy)(r)}{(1+x+ry)^2}$$

$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{q-pr+q\bar{y}}{(1+\bar{y}+r\bar{y})^2}$$

$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{q-pr+q\bar{y}+qr\bar{y}-qr\bar{y}}{(1+\bar{y}+r\bar{y})^2}$$

$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{q+q\bar{y}-r(p+q\bar{y})+qr\bar{y}}{(1+\bar{y}+r\bar{y})^2}$$

$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{q(1+\bar{y}+r\bar{y})}{(1+\bar{y}+r\bar{y})^2} - \frac{r(p+q\bar{y})}{(1+\bar{y}+r\bar{y})^2}$$

$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{q}{1+\bar{y}+r\bar{y}} - \frac{r\bar{y}}{1+\bar{y}+r\bar{y}}$$

$$\frac{\partial f}{\partial y}(\bar{y},\bar{y}) = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}.$$

The linearized equation is

$$y_{n+1} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}y_n - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}y_{n-k} = 0$$

and the characteristic equation is

$$\lambda^{n+1} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^n - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^{n-k} = 0,$$

which is equivalent to

$$\lambda^{k+1} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^k - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = 0.$$

3.4 Period two cycles

In particular, the solution $\{y_n\}_{n=-k}^{\infty}$ has a prime period two if it takes the form

$$\ldots \phi, \psi, \phi, \psi, \phi, \ldots$$

where ϕ, ψ are distinct and positive.

Theorem 3.1. *[5] Consider equation (3.2.1). Let k be an even integer. Then (3.2.1) has no non-negative distinct prime period two solution.*

Proof: Let k be an even integer. Assume that (3.2.1) has a two periodic cycle $\{\phi, \psi\}$, where ϕ and ψ are distinct and positive solution. Note that since k is even integer, $x_{n-k} = x_n \neq x_{n+1}$. The two period cycle satisfies:

$$\phi = \frac{p+q\psi}{1+\psi+r\psi}$$

and

$$\psi = \frac{p+q\phi}{1+\phi+r\phi}.$$

Substitute ϕ into the equation of ψ, we get:

$$\psi = \frac{p+q\frac{p+q\psi}{1+\psi+r\psi}}{1+\frac{p+q\psi}{1+\psi+r\psi}+r(\frac{p+q\psi}{1+\psi+r\psi})}$$

$$\psi = \frac{p(1+\psi+r\psi)+q(p+q\psi)}{1+\psi+r\psi+p+q\psi+rp+rq\psi}$$

$$(1+r+q+qr)\psi^2 + (1+p+pr-p-pr-q^2)\psi - (p+qp) = 0.$$

That is equivalent to

$$(1+r)\psi^2 + (1-q)\psi - p = 0.$$

The roots of this quadratic equation are:

$$\psi = \frac{q-1 \pm \sqrt{(1-q)^2 + 4p(1+r)}}{2(1+r)}$$

but

$$\sqrt{(1-q)^2 + 4p(1+r)} > \mid q-1 \mid .$$

So

$$\psi = \frac{q-1 + \sqrt{(1-q)^2 + 4p(1+r)}}{2(1+r)}.$$

Similarly,

$$\phi = \frac{q-1 + \sqrt{(1-q)^2 + 4p(1+r)}}{2(1+r)}.$$

We have $\phi = \psi$ and this is a contradiction. So (3.2.1) has no non-negative prime period two solution. $\diamond$

Theorem 3.2. *[7] Let k be odd integer. If $q \leq 1$, then (3.2.1) has no non-negative prime period-two solution.*

Proof: Let k be any odd integer and $q \leq 1$. Assume that (3.2.1) has two periodic cycle, say $\{\phi, \psi\}$. Since we assume k is an odd integer, $x_{n+1} = x_{n-k} \neq x_n$. So1

$$\phi = \frac{p+q\phi}{1+\psi+r\phi}$$

and

$$\psi = \frac{p+q\psi}{1+\phi+r\psi}.$$

From the equation of ϕ we have

$$\phi(1+\psi+r\phi) = p+q\phi$$

or

$$r\phi^2 + \phi\psi + \phi = p+q\phi$$

and from the equation of ψ we get

$$r\psi^2 + \phi\psi + \psi = p + q\psi.$$

Subtract the last two equation, we have

$$r(\phi^2 - \psi^2) - q(\phi - \psi) + (\phi - \psi) = 0.$$

Since ϕ and ψ are distinct, we can divide on $\phi - \psi$, we get

$$\phi + \psi = \frac{q-1}{r}$$

we have a contradiction since we assume $q \leq 1$ which implies $\frac{q-1}{r} < 0$ but ϕ and ψ are non-negative and distinct. $\diamond$

Theorem 3.3. *[5] Let k be any odd integer. Then (3.2.1) has a non-negative prime period-two solution if and only if*

$$q > 1 \quad \textit{and} \quad (r-1)(q-1)^2 + 4pr^2 < 0$$

Proof: Let k be any odd integer. Assume that $q > 1$, $(r-1)(q-1)^2 + 4pr^2 < 0$ and (3.2.1) has a non-negative prime period-two solution say $\ldots, \phi, \psi, \phi, \psi, \ldots$ such that

$$\phi = \frac{p + q\phi}{1 + \psi + r\phi}$$

and

$$\psi = \frac{p + q\psi}{1 + \phi + r\psi}.$$

By simple calculations, we get

$$r\phi^2 + \phi\psi + \phi = p + q\phi \tag{3.4.1}$$

and

$$r\psi^2 + \phi\psi + \psi = p + q\psi. \tag{3.4.2}$$

Subtract (3.4.3) from (3.4.1), we get

$$\phi + \psi = \frac{q-1}{r}.$$

Take the square of both sides, we have

$$\phi^2 + 2\phi\psi + \psi^2 = (\frac{q-1}{r})^2$$

or

$$\phi^2 + \psi^2 = (\frac{q-1}{r})^2 - 2\phi\psi.$$

Now, add (3.4.1) and (3.4.3), we get

$$\phi\psi = \frac{2p + (q-1)(\phi+\psi) - r(\phi^2+\psi^2)}{2}$$

$$\phi\psi = \frac{2p + \frac{(q-1)^2}{r} - r((\frac{q-1}{r})^2 - 2\phi\psi)}{2}$$

$$\phi\psi = \frac{p}{1-r}.$$

Note that this is possible since $r - 1 < \frac{-4pr^2}{(q-1)^2} < 0$ which implies $r < 1$. Now, ϕ and ψ are the positive roots of the quadratic polynomial

$$m^2 - \frac{q-1}{r}m + \frac{p}{1-r}.$$

This quadratic equation has discriminant s which is equal $s = (\frac{q-1}{r})^2 - 4\frac{p}{1-r}$ which is equivalent to

$$s = \frac{(q-1)^2(r-1) + 4pr^2}{r^2(r-1)}.$$

Since we assume that $(r-1)(q-1)^2 + 4pr^2 < 0$ and $q > 1$ which implies $r < 1$, we get $s > 0$ and this proves that (3.2.1) has non-negative prime period-two solution under the given two conditions. $\diamond$

To study the stability of the two cycle $\{\phi, \psi\}$.

Let

$$v_n = y_{n-k}$$

and

$$z_n = y_n, \quad n = 0, 1, 2, \ldots$$

Equation(3.2.1) corresponds to the following second dimensional system

$$v_{n+1} = z_n$$
$$z_{n+1} = \frac{p+qv_n}{1+z_n+rv_n}, \quad n = 0,1,2,\ldots.$$

Let F be the function on $(0,\infty)\times(0,\infty)$ defined by:

$$F\begin{pmatrix} v \\ z \end{pmatrix} = \begin{pmatrix} z \\ \frac{p+qv}{1+z+rv} \end{pmatrix},$$

then

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

is a fixed point of $F^2(v,z)$ where

$$F^2\begin{pmatrix} v \\ z \end{pmatrix} = \begin{pmatrix} F_1(v,z) \\ F_2(v,z) \end{pmatrix}$$

and $F_1(v,z) = \frac{p+qv}{1+z+rv}$ and $F_2(v,z) = \frac{p+qz}{1+F_1(v,z)+rz}$.

The two cycle $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$ is locally asymptotically stable if the eigenvalues of the Jacobian matrix JF^2 evaluated at $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$, lie inside the unit disk.

$$JF^2 = \begin{pmatrix} \frac{\partial F_1}{\partial v}(\phi,\psi) & \frac{\partial F_1}{\partial z}(\phi,\psi) \\ \frac{\partial F_2}{\partial v}(\phi,\psi) & \frac{\partial F_2}{\partial u}(\phi,\psi) \end{pmatrix}$$

$$\frac{\partial F_1}{\partial v}(v,z) = \frac{q-pr+qz}{(1+z+rv)^2},$$
$$\frac{\partial F_1}{\partial v}(\phi,\psi) = \frac{q-pr+q\psi}{(1+\psi+r\phi)^2},$$
$$\frac{\partial F_1}{\partial z}(v,z) = -\frac{p+qv}{(1+z+rv)^2}$$
$$\frac{\partial F_1}{\partial z}(\phi,\psi) = -\frac{p+q\phi}{(1+\psi+r\phi)^2},$$
$$\frac{\partial F_2}{\partial v}(v,z) = -\frac{(p+qz)\frac{\partial F_1}{\partial v}(v,z)}{(1+F_{1(v,z)+rz)^2}}.$$

Since $F_1(\phi,\psi) = \phi$

$$\frac{\partial F_2}{\partial v}(\phi,\psi) = -\frac{(p+q\psi)(q+q\psi-pr)}{(1+\psi+r\phi)^2(1+\phi+r\psi)^2},$$

$$\frac{\partial F_2}{\partial u}(v,u) = \frac{q(1+F_1(v,u)+rz)-(p+qz)(\frac{\partial F_1}{\partial v}(v,z)+r)}{(1+F_1(v,u)+rz)^2}$$

$$\frac{\partial F_2}{\partial u}(\phi,\psi) = \frac{q(1+\phi+r\psi)-(p+q\psi)(-\frac{p+q\phi}{(1+\psi+r\phi)^2}+r)}{(1+\phi+r\psi)^2}$$

$$\frac{\partial F_2}{\partial u}(\phi,\psi) = \frac{(p+q\psi)(p+q\phi)}{(1+\phi+r\psi)^2(1+\psi+r\phi)^2} + \frac{q+q\phi-pr}{(1+\phi+r\psi)^2}.$$

We want to check if the eigenvalues of $JF^2(\phi,\psi)$ lie inside the unit circle. Let T denote the trace of $JF^2(\phi,\psi)$ and D denotes the determinant of $JF^2(\phi,\psi)$
By theorem (1.7), it suffices to show that $\mid T \mid < 1 + D < 2$. That is equivalent to

$$D < 1 \tag{3.4.3}$$

$$T < 1 + D \tag{3.4.4}$$

$$-1 - D < T \tag{3.4.5}$$

To prove (3.4.3) we must show

$$\frac{(q+q\psi-pr)(q+q\phi-pr)}{(1+\phi+r\psi)^2(1+\psi+r\phi)^2} < 1$$

Let $(q+q\psi-pr)(q+q\phi-pr)$ be term 1 and $(1+\phi+r\psi)^2(1+\psi+r\phi)^2$ be term 2. Observe that

$$(q+q\psi-pr)(q+q\phi-pr) = (q-pr)^2 + q(\phi+\psi)(q-pr) + q^2\phi\psi.$$

Recall that $\phi+\psi = \frac{q-1}{r}$ and $\phi\psi = \frac{p}{1-r}$, so

$$(q+q\psi-pr)(q+q\phi-pr) = \frac{-2rq^2+r^2q^263pr^2q-2pr^3q-p^2r^3+p^2r^4-q^3+rq^3+q^2-pr^2q^2-qrp}{r(r-1)}$$

and

$$(1+\phi+r\psi)^2(1+\psi+r\phi)^2 = (1+\phi+r\psi+\psi+r\psi^2+r\phi+\phi^2r+r^2\phi\psi)^2$$

$$(1+\phi+r\psi)^2(1+\psi+r\phi)^2 = (\frac{-pr^2+qr+pr-q+q^2}{r})^2.$$

Subtract term 2 from term 1, we get

$$\frac{-pr^3q^2+2q^2r-q^2r^2-q^2+3p^2r^3-2p^2r^4-3rq^3+3qpr^3-5qpr^2}{r^2(1-r)}$$

$$+\frac{2prq+4r^2pq^2+2q^3+rq^4-r^2p^2-2prq^2-q^4+r^2q^3}{r^2(1-r)}$$

$$=\frac{(2r^2p-rp+q-q^2-rq+q^2r)(rp-r^2p-q+q^2+qr)}{r^2(1-r)}.$$

But since $q \geq 1$ and $r-1<0$, we get

$$rp-r^2p-q+q^2+qr = rp(1-r)+q(q-1)+qr > 0$$

and

$$2r^2p-rp+q-q^2-rq+q^2r < 2r^2p-rp+q-qr+2q(r-1)-(r-1)-4pr^2.$$

From $(r-1)(q-1)^2+4pr^2<0$, we note that $r-1<0$. So

$$= -2pr^2-rp+(r-1)(q-1) < 0.$$

That shows $D-1<0$ which is the first inequality.

Inequality (3.4.4) is equivalent to

$$(q-pr+q\psi)(1+\phi+r\psi^2)+(p+q\psi)(p+q\phi)+(q-pr+q\phi)(1+\psi+r\phi)^2$$

$$+(q-pr+q\phi)(q-pr+q\psi)+(1+\phi+r\psi)^2(1+\psi+r\varphi)^2 < 0.$$

Substitute $\phi+\psi = \frac{q-1}{r}$ and $\phi\psi = \frac{p}{1-r}$ and use the assumption $q>1$ and $(r-1)(q-1)^2+4pr^2<0$, after long calculations one can get

$$(q-pr+q\psi)(1+\phi+r\psi^2)+(p+q\psi)(p+q\phi)+(q-pr+q\phi)(1+\psi+r\phi)^2$$

$$+(q-pr+q\phi)(q-pr+q\psi)+(1+\phi+r\psi)^2(1+\psi+r\varphi)^2 < 0$$

This proves inequality (3.4.4).

The final inequality is equivalent to

$$(q-pr+q\psi)(1+\phi+r\psi)^2+(p+q\phi)(p+q\psi)+(q-pr+q\phi)(1+\psi+r\phi)^2$$

$$+(q-pr+q\phi)(q-pr+q\psi)+(1+\psi+q\phi)^2(1+\phi+q\psi)^2>0.$$

Similarly, one can show that the previous inequality is a consequence of our assumption.

This proves $T+D+1>0$.

So $\begin{pmatrix}\phi\\ \psi\end{pmatrix}$ is a locally stable fixed point of $F^2(v,z)$ and hence ϕ,ψ is a locally stable prime period-two cycle for (3.2.1) with odd integer k.[7]

3.5 Invariant Intervals

Definition 13. *An interval S is an invariant interval of the difference equation (3.2.1) if $y_{N-k},\ldots,y_{N-1},y_N\in S$ for some integer number N, then $y_n\in S$ for all $n\geq N$.*

Theorem 3.4. *[7] Consider the difference equation (3.2.1) with fixed integer k and $\{y_n\}_{n=-k}^{\infty}$ as a solution. Then we have the following invariant intervals:*

1. $[0,b]$ *when* $pr\leq q$ *where* $b=\frac{q-1+\sqrt{(q-1)^2+4pr}}{2r}$.
2. $[\frac{pr-q}{q},\frac{q}{r}]$ *when* $1<q<pr<q+\frac{q^2}{r}-\frac{q}{r}$.
3. $[\frac{q}{r},\frac{pr-q}{q}]$ *when* $pr\geq q+\frac{q^2}{r}$.
4. $[0,\frac{q}{r}]$ *when* $pr\leq q$.

Proof:(1) Let $b=\frac{q-1+\sqrt{(q-1)^2+4pr}}{2r}$. Suppose $pr\leq q$ and assume that $y_{N-k},\ldots,y_{N-1},y_N\in[0,b]$ for some integer number N.

Take the function

$$g(x)=\frac{p+qx}{1+rx}$$

$$\acute{g}(x)=\frac{q(1+rx)-r(p+qx)}{(1+rx)^2}=\frac{q-pr}{(1+rx)^2}.$$

By assumption that $pr \leq q$,

$$\acute{g} \geq 0$$

and

$$y_{N+1} = \frac{p + qy_{N-k}}{1 + y_N + ry_{N-k}} \leq \frac{p + qy_{N-k}}{1 + ry_{N-k}} = g(y_{N-k})$$

but

$$g(y_{N-k}) \leq g(b) \leq b$$

That shows $y_{N+1} \in [0, b]$. The proof follows by induction.

(2)Assume that $1 < q < pr < q + \frac{q^2}{r} - \frac{q}{r}$ and $y_{N-k}, \ldots, y_{N-1}, y_N \in [\frac{pr-q}{q}, \frac{q}{r}]$ for some integer N.

Let

$$f(x, y) = \frac{p + qy}{1 + x + ry}.$$

$$\frac{\partial f}{\partial y} = \frac{q - p - qx}{(1 + x + ry)^2}.$$

Note that $\frac{\partial f}{\partial y} > 0$ when $x \geq \frac{pr-q}{q}$. So f(x, y) is increasing in y for $x \geq \frac{pr-q}{q}$.

$$y_{N+1} = \frac{p + qy_{N-k}}{1 + y_N + ry_{N-k}} = f(y_N, y_{N-k})$$

$$y_{N+1} \geq f(\frac{q}{r}, \frac{pr - q}{q}) = \frac{p + pr - q}{1 + \frac{pr^2}{q} - r + \frac{q}{r}}$$

$$y_{N+1} > \frac{p + pr - q}{q + \frac{q}{r}} > \frac{pr - q}{q + \frac{q}{r}} > \frac{pr - q}{q}.$$

Also

$$y_{N+1} = f(y_N, y_{N-k}) \leq f(\frac{pr - q}{q}, \frac{q}{r}) = \frac{p + \frac{q^2}{r}}{1 + \frac{pr}{q} - 1 + r\frac{q}{r}} = \frac{q}{r}(\frac{\frac{pr}{q} + q}{\frac{pr}{q} + q}) = \frac{q}{r}.$$

The proof follows by induction.

(3)Assume that $pr \geq q + \frac{q^2}{r}$ and $y_{N-k}, \ldots, y_{N-1}, y_N \in [\frac{q}{r}, \frac{pr-q}{q}]$.

Note that $f(x, y) = \frac{p+qy}{1+x+ry}$ is decreasing in y for $x \leq \frac{pr-q}{q}$ since $\frac{\partial f}{\partial y} = \frac{q-p-qx}{(1+x+ry)^2}$.

$$y_{N+1} = \frac{p + qy_{N-k}}{1 + y_N + ry_{N-k}} = f(y_N, y_{N-k})$$

$$y_{N+1} \geq f(\frac{pr-q}{q}, \frac{pr-q}{q}) = \frac{p+pr-q}{1+\frac{pr}{q}-1+\frac{r}{q}(pr-q)}$$

$$y_{N+1} \geq \frac{pr+p-q}{\frac{r}{q}(p+pr-q)} = \frac{q}{r}.$$

Also

$$y_{N+1} = f(y_N, y_{N-k}) \leq f(\frac{q}{r}, \frac{q}{r}) = \frac{pr+q^2}{r+q+qr} < \frac{pr-q}{q}.$$

The proof follows by induction.

(4)Assume that $pr \leq q$ and $y_{N-k}, \ldots, y_{N-1}, y_N \in [0, \frac{q}{r}]$.

$$y_{N+1} = \frac{p+qy_{N-k}}{1+y_N+ry_{N-k}}$$

$$y_{N+1} = \frac{q(\frac{p}{q}+y_{N-k})}{r(\frac{1}{r}+\frac{1}{r}y_N+y_{N-k})}.$$

Since $pr \leq q$,

$$y_{N+1} \leq \frac{q(\frac{1}{r}+y_{N-k})}{r(\frac{1}{r}+\frac{1}{r}y_N+y_{N-k})}$$

$$\leq \frac{q(\frac{1}{r}+y_{N-k})}{r(\frac{1}{r}+y_{N-k})} = \frac{q}{r}$$

The proof follows by induction. $\diamond$

3.6 Boundedness

We will study the boundedness of the solution of (3.2.1)

Theorem 3.5. *[5] Every solution of (3.2.1) is bounded.*

Proof: Let $\{y_n\}_{n=-k}^{\infty}$ be a solution of (3.2.1). We need to show that the solution is bounded from above and from below.

If the solution is bounded from above by some constant M, then it is bounded from below since

$$y_{n+1} \geq \frac{p}{1+M+rM}, n = -k, -k+1, \ldots.$$

We will use contradiction to show that the solution is bounded from above. Assume not. So we can find a subsequence $\{y_{n_m}\}_{m=0}^{\infty}$ such that

$$m \to \infty, n_m \to \infty, y_{n_m+1} \to \infty$$

where

$$y_{n_m+1} = max\{y_n : n \leq n_m\}, \quad m \geq 0.$$

But (3.2.1) implies

$$y_{n+1} < p + qy_{n-k}, \quad n \geq 0.$$

So

$$\lim_{m\to\infty} y_{n_m+1} = \lim_{m\to\infty} y_{n_m-k} = \infty.$$

Take sufficiently large m

$$0 \leq y_{n_m+1} - y_{n_m-k} = \frac{p + qy_{n_m-k}}{1 + y_{n_m} + ry_{n_m-k}} - y_{n_m-k}$$

$$= \frac{p + [(q-1) - y_{n_m} - ry_{n_m-k}]y_{n_m-k}}{1 + y_{n_m} + ry_{n_m-k}} < 0$$

and this is a contradiction. This proves that the solution is bounded from above.

3.7 Global Stability

In this section we investigate the global stability of the positive equilibrium point of (3.2.1). While we do that , we need some results.

Lemma 3.6. *[7] Let $[a, b]$ be an interval of real numbers and assume that $f : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties:*

1. *$f(x, y)$ is non-increasing in $x \in [a, b]$ for each $y \in [a, b]$ and $f(x, y)$ is non-decreasing in $y \in [a, b]$ for all $x \in [a, b]$,*

2. *The difference equation has no solutions of prime period-two in $[a, b]$.*

Then (3.2.1) has a unique equilibrium $\bar{y} \in [a, b]$ and every solution converges to $\bar{y}$.

Theorem 3.7. *Assume that*

$$pr \leq q \text{ and } \ r \geq 1 \tag{3.7.1}$$

Then the equilibrium point of (3.2.1) $\bar{y}$ is a global attractor of all non-negative solutions of (3.2.1).

Proof: We have shown in theorem (3.4) that in this case $[0, \frac{q}{r}]$ is an invariant interval and all non-negative solutions of equation (3.2.1) lie in this interval since $0 \leq y_n \leq \frac{q}{r}$ for $n = 1, 2, \ldots$.
Consider the function

$$f(x, y) = \frac{p + qy}{1 + x + ry}$$

.

$$\frac{\partial f}{\partial x}(x, y) = -\frac{p + qy}{(1 + x + ry)^2} \ \ and \ \ \frac{\partial f}{\partial y}(x, y) = \frac{q + qx - pr}{(1 + x + ry)^2}.$$

So $f(x, y)$ is decreases in x and increases in y in $[0, \frac{q}{r}]$. The assumption $r \geq 1$ implies

$$(r - 1)(q - 1)^2 + 4pr^2 > 0.$$

By theorem (3.3), equation (3.2.1) has no solution of prime period two in $[0, \frac{q}{r}]$.
So if we have two positive real numbers m and M in $[0, \frac{q}{r}]$ which satisfies the following equations:

$$M = \frac{p + qM}{1 + m + rM}, \quad m = \frac{p + qm}{1 + M + rm}$$

Then $m = M$.
Lemma (3.6) implies that (3.2.1) has a unique equilibrium point $\bar{y} \in [0, \frac{q}{r}]$ and every solution of equation (3.2.1) converges to $\bar{y}$. Thus $\bar{y}$ is global attractor of non-negative solution of (3.2.1). $\diamond$

Theorem 3.8. *[5] Assume either*

$$q \leq 1 \quad or \quad q > 1, (r - 1)(q - 1)^2 + 4pr^2 > 0.$$

Then the equilibrium point $\bar{y}$ is globally asymptotically stable.

3.8 Numerical Discussion

Example 3.1. *Assume equation (3.2.1) holds. Take* $k = 3, p = 0.8; q = 1.5$ *and* $r = 0.5$. *Equation (3.2.1) becomes*

$$y_{n+1} = \frac{0.8 + 1.5y_{n-3}}{1 + y_n + 0.5y_{n-3}}, n = 0, 1, 2, \ldots$$

Consider the positive initial conditions are $y_0 = 0.1, y_1 = 1.1, y_2 = 0.2$ *and* $y_3 = 1$. *The positive equilibrium point is*

$$\bar{y} = \frac{1.5 - 1 + \sqrt{(1 - 1.5)^2 + 4(0.8)(1 + 0.5)}}{2(1 + 0.5)} = 0.91574017.$$

Theoretically $\bar{y}$ *is stable since* $q = 1.5 > 1$ *and* $(r - 1)(1 - q)^2 + 4pr^2 = 0.675 > 0$. *Figure (3.2) show that* $lim_{n\to\infty} y_n = \bar{y}$. *This also shows* $\bar{y}$ *is globally asymptotically stable.*

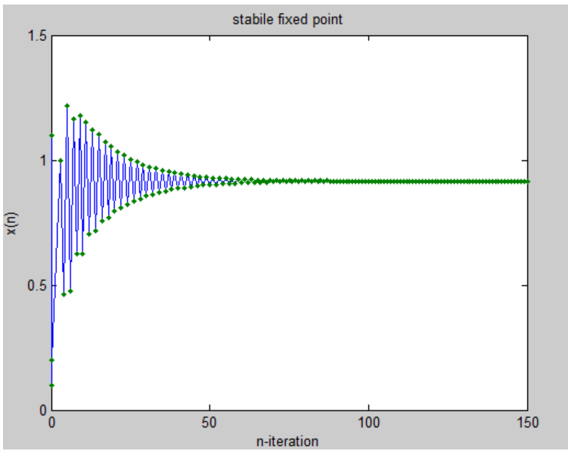

Fig. 3.1: Figure shows that the fixed point is globally asymptotically stable.

Example 3.2. *Assume equation (3.2.1) holds. Take* $k = 1, p = 4; q = 5$ *and* $r = 0.5$. *Equation (3.2.1) becomes*

$$y_{n+1} = \frac{4 + 5y_{n-1}}{1 + y_n + 0.5y_{n-1}}, n = 0, 1, 2, \ldots$$

Consider the positive initial conditions are $y_0 = 0.1$ and $y_1 = 1.1$.
The equilibrium point is

$$\bar{y} = \frac{5-1+\sqrt{(1-5)^2+4(4)(1+0.5)}}{2(1+0.5)} = 3.44154843$$

This fixed point is unstable since $q = 5 > 1$ and $(r-1)(1-q)^2 + 4pr^2 = -4 < 0$. Figure (3.1) shows that $\bar{y}$ is unstable.

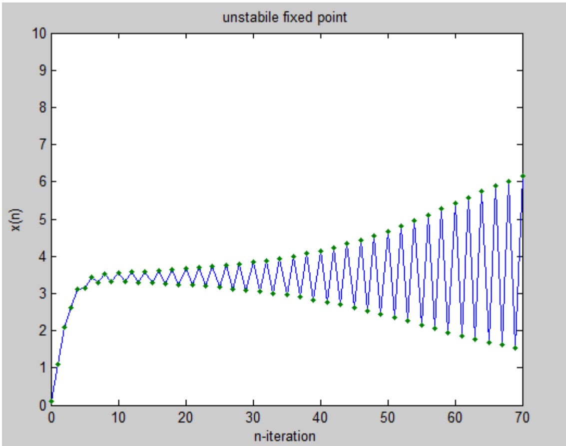

Fig. 3.2: The equilibrium point is unstable.

4

Dynamics And Bifurcation Of $y_{n+1} = \frac{p+qy_{n-1}}{1+y_n+ry_{n-1}}$

In this chapter we study dynamics and bifurcation of the positive fixed point of the nonlinear second order rational difference equation

$$y_{n+1} = \frac{p + qy_{n-1}}{1 + y_n + ry_{n-1}}, n = 0, 1, 2, \ldots \tag{4.0.1}$$

We consider the saddle-node, period-doubling and Neimark-Sacker bifurcation.

4.1 Dynamics of $y_{n+1} = \frac{p+qy_{n-1}}{1+y_n+ry_{n-1}}$

In this section we study the stability of the fixed point of equation (4.0.1). Recall that the discrete difference equation (4.0.1) has the unique positive fixed point

$$\bar{y} = \frac{q - 1 + \sqrt{(1-q)^2 + 4p(1+r)}}{2(1+r)}.$$

In order to convert equation (4.0.1) to a second dimensional system with three positive parameters p, q, and r, let $u_n = x_{n-1}$ and $w_n = x_n$. We have the following system

$$\begin{aligned} u_{n+1} &= w_n \\ w_{n+1} &= \frac{p + qu_n}{1 + w_n + ru_n}, n = 0, 1, 2, \ldots. \end{aligned} \tag{4.1.1}$$

System (4.1.1) has the unique positive fixed point $(u^*, w^*)^T = (\bar{y}, \bar{y})^T$. Convert this system into second dimensional map

$$F\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} f_1(u,w) \\ f_2(u,w) \end{pmatrix} = \begin{pmatrix} w \\ \frac{p+qu}{1+w++ru} \end{pmatrix} \tag{4.1.2}$$

We need to find the Jacobian of $F(u, w)$. Note that

$$\frac{\partial f_1}{\partial u} = 0,$$

$$\frac{\partial f_1}{\partial w} = 1,$$

$$\frac{\partial f_2}{\partial u} = \frac{q+qw-rp}{(1+w+ru)^2}$$

and

$$\frac{\partial f_2}{\partial w} = -\frac{p+qu}{(1+w+ru)^2}.$$

The Jacobian matrix is

$$JF(u,w) = \begin{pmatrix} 0 & 1 \\ \frac{q+qw-rp}{(1+w+ru)^2} & -\frac{p+qu}{(1+w+ru)^2} \end{pmatrix}$$

$$JF(u,w)\mid_{(\bar{y},\bar{y})} = \begin{pmatrix} 0 & 1 \\ \frac{q+q\bar{y}-rp}{(1+\bar{y}+r\bar{y})^2} & -\frac{p+q\bar{y}}{(1+\bar{y}+r\bar{y})^2} \end{pmatrix}$$

Note that in chapter 3 we show that

$$\det(JF(\bar{y},\bar{y})) = -\frac{q+q\bar{y}-rp}{(1+\bar{y}+r\bar{y})^2} = -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$$

and

$$tr(JF(\bar{y},\bar{y})) = -\frac{p+q\bar{y}}{(1+\bar{y}+r\bar{y})^2} = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}}$$

where det and tr denote the determinant and trace of the Jacobian matrix J, respectively.

Substitute the value of $\bar{y}$, we have

$$\det(JF(\bar{y},\bar{y})) = -\frac{q - r\frac{q-1+\sqrt{(1-q)^2+4p(1+r)}}{2(1+r)}}{1+(1+r)(\frac{q-1+\sqrt{(1-q)^2+4p(1+r)}}{2(1+r)})} = \frac{-2q-qr-r+r\sqrt{(1-q)^2+4p(1+r)}}{(1+r)(q+1+\sqrt{(1-q)^2+4p(1+r)})},$$

$$tr(JF(\bar{y},\bar{y})) = -\frac{\frac{q-1+\sqrt{(1-q)^2+4p(1+r)}}{2(1+r)}}{1+(1+r)(\frac{q-1+\sqrt{(1-q)^2+4p(1+r)}}{2(1+r)})} = \frac{1-q-\sqrt{(1-q)^2+4p(1+r)}}{(1+r)(q+1+\sqrt{(1-q)^2+4p(1+r)})}.$$

Theorem (1.7) implies that the fixed point $(\bar{y},\bar{y})^T$ is asymptotically stable if the following inequality holds

$$\mid tr(J) \mid -1 < det(J) < 1 \tag{4.1.3}$$

where J is the Jacobian matrix evaluated at this fixed point.

Theorem 4.1. *[5] The equilibrium point $\bar{y}$ of (4.0.1) is locally asymptotically stable if one of the following holds*

1. $q \leq 1$
2. $q > 1$ *and* $(r-1)(q-1)^2 + 4pr^2 > 0$.

Proof: We want to show that

$$| \frac{q - r\bar{y}}{1+\bar{y}+r\bar{y}} |< 1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}} < 2$$

That is equivalent to

$$\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + | \frac{q - r\bar{y}}{1+\bar{y}+r\bar{y}} |< 1 \quad and \quad \frac{\bar{y}}{1+\bar{y}+r\bar{y}} > -1.$$

The first inequality is equivalent to

$$| q - r\bar{y} |< 1 + r\bar{y}. \tag{4.1.4}$$

If $q - r\bar{y} < 0$, then(4.1.4) becomes $r\bar{y} - q < 1 + r\bar{y}$ and this is obvious .
If $q - r\bar{y} \geq 0$, then(4.1.4) becomes $q - r\bar{y} < 1 + r\bar{y}$
or

$$q - 1 < 2r\bar{y}. \tag{4.1.5}$$

If $q \leq 1$, then (4.1.5) holds. If $q > 1$, then

$$r\bar{y} > r\frac{\sqrt{(q-1)^2 + 4p(1+r)}}{r+1} > r\sqrt{(q-1)^2 + 4p(1+r)}$$

and if $(r-1)(q-1)^2 + 4pr^2 > 0$, multiply both sides by $r+1$ we can get

$$(r^2 - 1)(q-1)^2 + 4pr^2(1+r) > 0.$$

Rearrange the terms of the previous inequality, we get

$$r^2((q-1)^2 + 4p(1+r)) > (q-1)^2.$$

Take the square of both sides, we obtain

$$r\sqrt{(q-1)^2+4p(1+r)} > (q-1).$$

Now, add $r(q-1)$ for both sides, we have

$$r(q-1+\sqrt{(q-1)^2+4p(1+r)} > (r+1)(q-1).$$

That is equivalent to

$$2r(r+1)\bar{y} > (r+1)(q-1)$$

or

$$2r\bar{y} > q-1.$$

This shows in this case inequality (4.1.5) holds and hence

$$\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + \mid \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \mid < 1.$$

Note that the second inequality $1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}} < 2$ is always true.
So in both cases the equilibrium point $\bar{y}$ is locally asymptotically stable.

4.2 Bifurcation of $y_{n+1} = \frac{p+qy_{n-1}}{1+y_n+ry_{n-1}}$

Inequality (4.1.3) is equivalent to the following three inequalities

- $\det(J) < 1$
- $\det(J) > tr(J) - 1$
- $\det(J) > -tr(J) - 1$

These three inequalities determine the stable region of $(\bar{y}, \bar{y})^T$ in the plane.
Saddle-node bifurcation happens in the trace-determinant plane when $\det(J) =$

$tr(J) - 1$. So the fixed point $(\bar{y}, \bar{y})^T$ of the system (4.1.2) undergoes a saddle-node bifurcation if

$$\det(JF(\bar{y}, \bar{y})) = tr(JF(\bar{y}, \bar{y})) - 1$$

or

$$\frac{-2q - qr - r + r\sqrt{(1-q)^2 + 4p(1+r)}}{(1+r)(q+1+\sqrt{(1-q)^2+4p(1+r)})} = \frac{1 - q - \sqrt{(1-q)^2 + 4p(1+r)}}{(1+r)(q+1+\sqrt{(1-q)^2+4p(1+r)})} - 1.$$

That is equivalent to

$$-2q - qr - r + r\sqrt{(1-q)^2 + 4p(1+r)} = 1 - q - \sqrt{(1-q)^2 + 4p(1+r)} - q - 1$$
$$-\sqrt{(1-q)^2 + 4p(1+r)} - qr - r - r\sqrt{(1-q)^2 + 4p(1+r)}$$

or

$$(2r+2)\sqrt{(1-q)^2 + 4p(1+r)} = 0.$$

That implies $2r + 2 = 0$ or $(1-q)^2 + 4p(1+r) = 0$.
If $2r + 2 = 0$, then $r = -1$ and this is impossible since r is a positive parameter.
If $(1-q)^2 + 4p(1+r) = 0$, then $p = -\frac{(1-q)^2}{4(1+r)} < 0$ and this is also impossible since p is positive parameter.

Theorem 4.2. *The fixed point $(\bar{y}, \bar{y})^T$ of the system (4.1.2) undergoes a period-doubling (flip) bifurcation when $p = \frac{(1-r)(q-1)^2}{4r^2}$ if $q > 1$.*

Proof: Assume that $q > 1$. Period doubling bifurcation occurs if $\det(JF(\bar{y}, \bar{y})^T) = -tr(JF(\bar{y}, \bar{y})) - 1$.
That is equivalent to

$$\frac{-2q - qr - r + r\sqrt{(1-q)^2 + 4p(1+r)}}{(1+r)(q+1+\sqrt{(1-q)^2+4p(1+r)})} = -\frac{1 - q - \sqrt{(1-q)^2 + 4p(1+r)}}{(1+r)(q+1+\sqrt{(1-q)^2+4p(1+r)})} - 1$$

or

$$-2q - qr - r + r\sqrt{(1-q)^2 + 4p(1+r)} = -1 + q + \sqrt{(1-q)^2 + 4p(1+r)} - q - qr - r - 1$$
$$-\sqrt{(1-q)^2 + 4p(1+r)} - r\sqrt{(1-q)^2 + 4p(1+r)}.$$

That implies

$$-2q + 2 + 2r\sqrt{(1-q)^2 + 4p(1+r)} = 0$$

Divide both sides by 2, we have

$$-q + 1 + r\sqrt{(1-q)^2 + 4p(1+r)} = 0.$$

Rearranging the terms, we get

$$r\sqrt{(1-q)^2 + 4p(1+r)} = q - 1. \tag{4.2.1}$$

and this is possible since we assume $q > 1$.
Take the square of both sides of equation (4.2.1), we get

$$r^2[(1-q)^2 + 4p(1+r)] = (q-1)^2$$

or

$$(r^2 - 1)(q-1)^2 + 4pr^2(1+r) = 0.$$

Since $r > 0$, $r + 1 \neq 0$ so we can divide into $1 + r$. We obtain

$$(r-1)(q-1)^2 + 4pr^2 = 0$$
$$p = \frac{(1-r)(q-1)^2}{4r^2}.$$

Note that the fixed point $(\bar{y}, \bar{y})^T$ is asymptotically stable if $q > 1$ and $(r-1)(q-1)^2 + 4pr^2 > 0$ and unstable if $q > 1$ and $(r-1)(q-1)^2 + 4pr^2 < 0$.

Now we consider the Neimark-Sacker bifurcation which is present when the Jacobian matrix has two conjugate eigenvalues with modulus one. This is corresponding to the case

$$det(J) = 1 \tag{4.2.2}$$

and

$$-2 < tr(J) < 2$$

on the trace-determinant plane.
Equation (4.2.2) holds if

$$\frac{-2q - qr - r + r\sqrt{(1-q)^2 + 4p(1+r)}}{(1+r)(q + 1 + \sqrt{(1-q)^2 + 4p(1+r)})} = 1.$$

That is equivalent to

$$-2q - qr - r + r\sqrt{(1-q)^2 + 4p(1+r)} = q + qr + 1 + r + \sqrt{(1-q)^2 + 4p(1+r)}$$
$$+r\sqrt{(1-q)^2 + 4p(1+r)}$$

or

$$-(3q+1) - 2r(q+1) = \sqrt{(1-q)^2 + 4p(1+r)}.$$

This can not happen since $q > 0$ and $r > 0$. So the system can not undergoes a Neimark-Sacker bifurcation at the fixed point $(\bar{y}, \bar{y})^T$

4.3 Direction of The Period-Doubling (Flip) Bifurcation

In this section we will find the direction of the period doubling bifurcation of system (4.1.1) at $p = \frac{(1-r)(q-1)^2}{4r^2}$. Firstly, we shift the fixed point $(\bar{y}, \bar{y})^T$ to the origin. Let

$$x_n = u_n - \bar{y}, \quad z_n = w_n - \bar{y}.$$

System (4.1.1) will be

$$x_{n+1} = z_n$$
$$z_{n+1} = \frac{p + q(x_n + \bar{y})}{1 + (z_n + \bar{y}) + r(x_n + \bar{y})} - \bar{y} \tag{4.3.1}$$

or

$$Y_{n+1} = AY_n + G(Y_n) \tag{4.3.2}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & -\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \end{pmatrix}, Y_n = \begin{pmatrix} x_n \\ z_n \end{pmatrix},$$

and

$$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(\| Y \|^3),$$

$$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ B_2(Y,Y) \end{pmatrix} \quad \textit{and} \quad C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ C_2(Y,Y,Y) \end{pmatrix}$$

where
$B_i(x,y) = \sum_{j,k=1}^{2} \frac{\partial^2 Y_i(\xi)}{\partial\xi_j\partial\xi_k} |_{\xi=0} (x_j y_k)$ and $C_i(x,y,z) = \sum_{j,k,l}^{2} \frac{\partial^3 Y_i(\xi)}{\partial\xi_j\partial\xi_k\partial\xi_l} |_{\xi=0} (x_j y_k z_l)$.

So $B_1(\phi,\psi) = 0$ and $C_1(\phi,\psi,\eta) = 0$,

$B_2(\phi,\psi) = -\frac{2r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}\phi_1\psi_1 + \frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}[\phi_1\psi_2 + \phi_2\psi_1] + 2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}\phi_2\psi_2$,

and $C_2(\phi,\psi,\eta) = 6\frac{r^2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3}\phi_1\psi_1\eta_1 + \frac{4qr-6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3}[\phi_1\psi_1\eta_2 + \phi_2\psi_0\eta_1 + \phi_1\psi_2\eta_1]$

$+\frac{2q-6r\bar{y}}{(1+\bar{y}+r\bar{y})^3}[\phi_1\psi_2\eta_2 + \phi_2\psi_1\eta_2 + \phi_1\psi_2\eta_2] - 6\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^3}\phi_2\psi_2\eta_2$.

Now, we find the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$ at the bifurcation point $p = \frac{(1-r)(q-1)^2}{4r^2}$. Recall that at this bifurcation point $2r\bar{y} = 1-q$. Let q and p^* be the eigenvectors of A and A^T corresponding to the eigenvalue $\lambda = -1$, respectively. We have $Aq = -q$ and $A^T p^* = -p^*$. The first equation can be written as $(A-\lambda I)q = (A+I)q = 0$
or

$$\begin{pmatrix} 1 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & 1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. Let $q_1 = 1$, the first equation $q_1 + q_2 = 0$ implies $q_2 = -1$. So take $q \sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Note that the vector above satisfies the second equation $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}q_1 + (1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}})q_2 = 0$ since at the bifurcation point $p = \frac{(1-r)(q-1)^2}{4r^2}$, $2r\bar{y} = 1-q$. Also in order to have a non zero solution of the system$(A+I)q = 0$, the matrix $A+I$ must be nonsingular. That means $\mid A+I \mid = 0$ or $1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = 0$.
The second equation can be written as $(A^T - \lambda I)p^* = (A^T + I)p^* = 0$ or

$$\begin{pmatrix} 1 & \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 & 1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}1} \end{pmatrix} \begin{pmatrix} p_1^* \\ p_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where $p^* = \begin{pmatrix} p_1^* \\ p_2^* \end{pmatrix}$. Take $p_2^* = 1$, the first equation $p_1^* + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}p_2^* = 0$ implies $p_1^* = -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$.

Note that this eigenvector $p^* \sim \begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 \end{pmatrix}$ satisfies the second equation

$$p_1^* + (1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}})p_2^* = 0.$$

Now, we normalize p^* and q.

$$< p^*, q >= \sum_{i=1}^{2} p_i^* q_i = -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - 1.$$

Take $p = \xi * \begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 \end{pmatrix}$ where $\xi = \frac{1}{-1-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}} = -\frac{1+\bar{y}+r\bar{y}}{q+1+\bar{y}}$.

The critical eigenspace T^c corresponding to the eigenvalue λ is a one-dimensional map and is spanned by the eigenvector q. Let T^{su} denote a one-dimensional linear eigenspace of A corresponding to the other eigenvalue than λ. Note that the matrix $A - \lambda I$ which is equivalent to the matrix $A + T$ has common invariant spaces with the matrix A, we conclude that $y \in T^{su}$ if and only if $< p, y >= 0$. Any vector $x \in \mathbb{R}^2$ can be decomposed as

$$x = uq + y$$

where $uq \in T^c, y \in T^{su}$ and

$$u =< p, x >$$

$$y = x- < p, x > q. \tag{4.3.3}$$

In the coordinates (u, y), the map (4.3.2) can be written as

$$\tilde{u} = \lambda u+ < p, F(uq + y) >,$$

$$\tilde{y} = Ay + F(uq + y)- < p, F(uq + y) > q. \tag{4.3.4}$$

Using Taylor expansions, (4.3.4) can be written as

$$\tilde{u} = \lambda u + \frac{1}{2}\sigma u^2 + u < b, y > + \frac{1}{6}\delta u^3 + \ldots,$$

$$\tilde{y} = Ay + \frac{1}{2}au^2 + \dots, \tag{4.3.5}$$

where $u \in \mathbb{R}^1, y \in \mathbb{R}^2, \sigma, \delta \in \mathbb{R}^1, a, b \in \mathbb{R}^2$ and $< b, y >= \sum_{i=1}^{2} b_i y_i$ is the standard scaler product $< b, y >$ can be expressed as

$$< b, y >=< p, B(q, y) > .$$

The center manifold of (4.3.5) has the representation

$$y = V(u) = \frac{1}{2}w_2u^2 + O(u^3),$$

where $w_2 \in T^{su} \subset \mathbb{R}^2$, so that $< p, w >= 0$. The vector w_2 satisfies

$$(A - I)w_2 + a = 0$$

Note that the matrix $A - I$ is invertible in $\mathbb{R}^2$ because $\lambda = 1$ is not an eigenvalue of A. Thus, we have

$$w_2 = -(A - I)^{-1}a$$

and the restriction of (4.3.5) to the center manifold takes the form

$$\tilde{u} = -u + \frac{1}{2}\sigma u^2 + \frac{1}{6}(\delta - 3 < p, B(q, (A - I)^{-1}a) >)u^3 + O(u^4)$$

where $\sigma =< p, B(q, q) >, \delta =< p, C(q, q, q) >$ and $a = B(q, q)- < p, B(q, q) > q$.

Using the identity $(A - I)^{-1}q = -\frac{1}{2}q$, the restricted map can be written as

$$\tilde{u} = -u + a(0)u^2 + b(0)u^3 + O(u^4), \tag{4.3.6}$$

where

$$a(0) = \frac{1}{2} < p, B(q, q) >$$

and

$$b(0) = \frac{1}{6} < p, C(q, q, q) > -\frac{1}{4}(< p, B(q, q) >)^2 - \frac{1}{2} < p, B(q, (A - I)^{-1}B(q, q)) > .$$

The map (4.3.6) can be transformed to the normal form

$$\tilde{\xi} = -\xi + c(0)\xi^3 + O(\xi^4)$$

where

$$c(0) = a^2(0) + b(0).$$

Thus, the critical normal form coefficient $c(0)$ allows us to predict the direction of bifurcation of period-two cycle. $c(0)$ is given by the following invariant formula:

$$c(0) = \frac{1}{6} < p, C(q,q,q) > -\frac{1}{2} < p, B(q,(A-I)^{-1}B(q,q)) > .$$

If $c(0) > 0$, then a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $p = \frac{(1-r)(1-q)^2}{4r^2}$.

$$B(q,q) = \begin{pmatrix} 0 \\ -2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2} \end{pmatrix},$$

$$< p, B(q,q) >= -\frac{1+\bar{y}+r\bar{y}}{q+1+\bar{y}}[-2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}],$$

$$C(q,q,q) = \begin{pmatrix} 0 \\ 6\frac{r^2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3} - 3\frac{4qr-6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3} + 3\frac{2q-6r\bar{y}}{(1+\bar{y}+r\bar{y})^3} + 6\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^3} \end{pmatrix},$$

$$< p, C(q,q,q) >= -\frac{1+\bar{y}+r\bar{y}}{q+1+\bar{y}}[6\frac{r^2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3} - 3\frac{4qr-6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3} + 3\frac{2q-6r\bar{y}}{(1+\bar{y}+r\bar{y})^3} + 6\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^3}],$$

$$(A-I)^{-1} = \begin{pmatrix} -1 & 1 \\ \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & -1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \end{pmatrix}^{-1} = \frac{1+\bar{y}+r\bar{y}}{2\bar{y}} \begin{pmatrix} -1-\frac{\bar{y}}{1+\bar{y}+r\bar{y}} & -1 \\ -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & -1 \end{pmatrix},$$

$$(A-I)^{-1}B(q,q) = \frac{1+\bar{y}+r\bar{y}}{2\bar{y}} \begin{pmatrix} -2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2} \\ -2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2+2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2}} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2} \end{pmatrix},$$

$$B(q,(A-I)^{-1}B(q,q)) = \frac{1+\bar{y}+r\bar{y}}{2\bar{y}} \begin{pmatrix} 0 \\ S \end{pmatrix}$$

where $S = [\frac{2r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2} + \frac{2\bar{y}}{(1+\bar{y}+r\bar{y})^2}][-2\frac{r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2} + 2\frac{\bar{y}}{(1+\bar{y}+r\bar{y})^2} - 2\frac{2r\bar{y}-q}{(1+\bar{y}+r\bar{y})^2}]$,

$$< p, B(q,(A-I)^{-1}B(q,q)) >= 2\frac{r^2(q-r\bar{y})^2}{\bar{y}(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2} - 2\frac{\bar{y}}{(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2} + 2\frac{r(q-r\bar{y})(2r\bar{y}-q)}{\bar{y}(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2} +$$

$2\frac{2r\bar{y}-q}{(q+1+\bar{y})(1+\bar{y}+r\bar{y})^2}$.

4.4 Numerical Discussion

In this section we give numerical examples which support our results in the previous sections. Figures that we get using Matlab will be attached with example to illustrate the bifurcation.

Example 4.1. *Consider equation (4.0.1). Fix the parameters p, r and consider q as bifurcation parameter.*
Take $p = 1$, $r = 0.9$ and $0 < q \leq 10$. Equation (4.0.1) becomes

$$y_{n+1} = \frac{1 + qy_{n-1}}{1 + y_n + 0.9y_{n-1}}, n = 0, 1, 2, \dots \tag{4.4.1}$$

The planer form corresponding to equation (4.4.3) is

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{1+qy_1(n)}{1+y_2(n)+0.9y_1(n)} \end{pmatrix} \tag{4.4.2}$$

Positive equilibrium point of system (4.4.4) is $(\bar{y}, \bar{y})$ where $\bar{y} = \frac{q-1+\sqrt{(1-q)^2+3.24}}{3.8}$. Theorem (4.2)determined the bifurcation point at $(r-1)(1-q)^2 + 4pr^2 = 0$. So the fixed point undergoes a period-doubling bifurcation at $q = 6.69209979$. See figure (4.1).

Example 4.2. *Consider equation (4.0.1). In this example we fix the parameters q, r and consider q as bifurcation parameter.*
Take $q = 1.1$, $r = 0.09$ and $0 < p \leq 2$. Equation (4.0.1) becomes

$$y_{n+1} = \frac{p + 1.1y_{n-1}}{1 + y_n + 0.09y_{n-1}}, n = 0, 1, 2, \dots \tag{4.4.3}$$

The planer form corresponding to equation (4.4.3) is

$$\begin{pmatrix} y_1(n+1) \\ y_2(n+1) \end{pmatrix} = \begin{pmatrix} y_2(n) \\ \frac{p+1.1y_1(n)}{1+y_2(n)+0.09y_1(n)} \end{pmatrix} \tag{4.4.4}$$

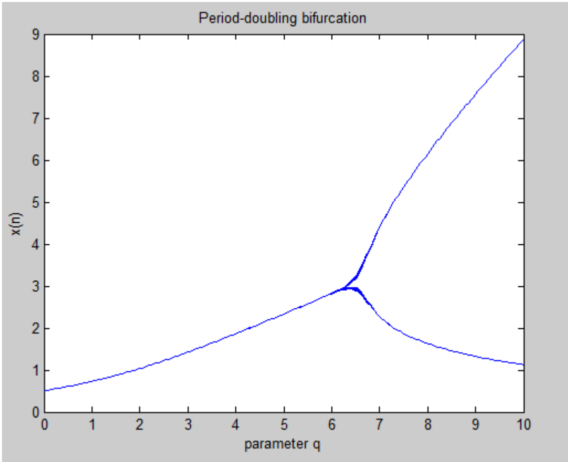

Fig. 4.1: Period-doubling bifurcation of the map $y_{n+1} = \frac{1+qy_{n-1}}{1+y_n+0.9yn-1}$, q is a parameter.

Positive equilibrium point of system (4.4.4) is $(\bar{y}, \bar{y})$ *where* $\bar{y} = \frac{0.1+\sqrt{0.01+4.36p}}{2.18}$. *Theorem (4.2)determined the bifurcation point at* $(r-1)(1-q)^2 + 4pr^2 = 0$. *So the fixed point undergoes a period-doubling bifurcation at* $p = 0.2808642$.

$$q = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad and \quad p = \begin{pmatrix} 0.39539749 \\ -0.60460251 \end{pmatrix},$$

$$B(q,q) = \begin{pmatrix} 0 \\ 0.71303782 \end{pmatrix},$$

$$< p, B(q,q) >= -0.43110446,$$

$$C(q,q,q) = \begin{pmatrix} 0 \\ -0.4797597 \end{pmatrix},$$

$$< p, C(q,q,q) >= 0.2900639,$$

$$(A-I)^{-1} = \begin{pmatrix} -1 & 1 \\ 0.65397924 & -1.34602076 \end{pmatrix}$$

$$B(q,(A-I)^{-1}B(q,q)) = \begin{pmatrix} 0 \\ 0.8212105 \end{pmatrix},$$

$$< p, B(q, (A-I)^{-1}B(q,q)) >= -0.49947486$$

$$c(0) = 0.20139345 > 0$$

So this verify that a unique and stable period-two cycle bifurcates from the fixed point at the bifurcation point $p = 0.2808642$. *See figure (4.1).*

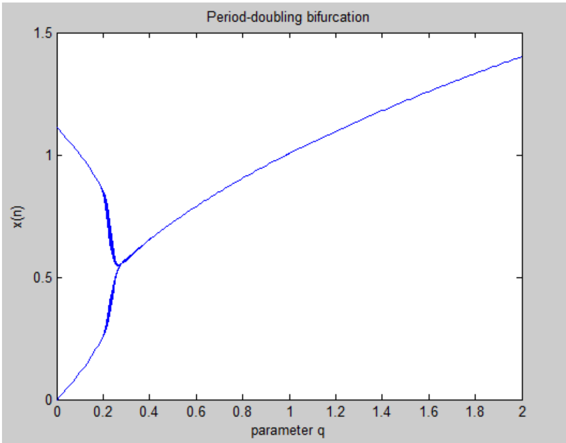

Fig. 4.2: Period-doubling bifurcation of $y_{n+1} = \frac{p+1.1y_{n-1}}{1+y_i+0.09y_{i-1}}$, p is a parameter.

5

Dynamics And Bifurcation Of $y_{n+1} = \frac{p+qy_{n-2}}{1+y_n+ry_{n-2}}$

In this chapter we will study the dynamics of the third order rational difference equation

$$y_{n+1} = \frac{p + qy_{n-2}}{1 + y_n + ry_{n-2}} \tag{5.0.1}$$

with positive parameters p, q and r and non-negative initial conditions y_{-2}, y_{-1} and y_0.

Then we will find the type of the bifurcation which exists at the point where the stability exchange.

5.1 Dynamics Of The Rational Difference Equation $y_{n+1} = \frac{p+qy_{n-2}}{1+y_n+ry_{n-2}}$

Consider equation (5.0.1). Equation (5.0.1) has the unique positive fixed point $\bar{y} = \frac{q-1+\sqrt{(q-1)^2+4p(1+r)}}{2(1+r)}$.

In order to convert equation (5.0.1) to a third dimensional system, let $z_n = y_n, x_n = y_{n-1}$ and $t_n = y_{n-2}$. We have the following system

$$z_{n+1} = \frac{p + qt_n}{1 + z_n + rt_n}$$

$$x_{n+1} = z_n$$

$$t_{n+1} = x_n \tag{5.1.1}$$

System (5.1.1) has the positive fixed point $(\bar{y}, \bar{y}, \bar{y})$. In order to shift this fixed point to the origin, let $w_n = z_n - \bar{y}$, $v_n = x_n - \bar{y}$ and $u_n = t_n - \bar{y}$. The system becomes

$$w_{n+1} = \frac{p + q(u_n + \bar{y})}{1 + (w_n + \bar{y}) + r(u_n + \bar{y})} - \bar{y}$$

$$v_{n+1} = w_n$$

$$u_{n+1} = v_n \tag{5.1.2}$$

System (5.1.2) has $(0,0,0)$ as a fixed point.

The Jacobian matrix of system (5.1.2) is

$$J(w,v,u) = \begin{pmatrix} -\frac{p+q(w_n+\bar{y})}{(1+w_n+\bar{y}+r(u_n+\bar{y}))^2} & 0 & \frac{q(1+w_n+\bar{y})-rp}{(1+w_n+\bar{y}+r(u_n+\bar{y}))^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J(0,0,0) = \begin{pmatrix} -\frac{p+q\bar{y}}{(1+\bar{y}+r\bar{y})^2} & 0 & \frac{q+q\bar{y}-rp}{(1+\bar{y}+r\bar{y})^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\bar{y}}{1+\bar{y}+r\bar{y}} & 0 & \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of the Jacobian matrix J is

$$p(\lambda) = -\lambda^3 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}. \tag{5.1.3}$$

Let $p_1 = \frac{\bar{y}}{1+\bar{y}+r\bar{y}}$, $p_2 = 0$ and $p_3 = -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$.

Theorem (1.8) implies that the zero solution is asymptotically stable if

$$\mid p_1 + p_3 \mid < 1 + p_2 \quad \textit{and} \quad \mid p_2 - p_1p_3 \mid < 1 - p_3^2. \tag{5.1.4}$$

Inequality (5.1.4) implies that the zero solution is asymptotically stable if

$$\mid \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \mid < 1 \tag{5.1.5}$$

and

$$\mid \frac{\bar{y}}{1+\bar{y}+r\bar{y}} \times \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} \mid < 1 - (\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})^2. \tag{5.1.6}$$

Inequality (5.1.5) is equivalent to

$$1 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > 0, \tag{5.1.7}$$

and

$$1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}} + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > 0, \tag{5.1.8}$$

and inequality (5.1.6) is equivalent to

$$1 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}} \times \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - (\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})^2 > 0, \tag{5.1.9}$$

and

$$1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}} \times \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - (\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})^2 > 0. \tag{5.1.10}$$

Inequality (5.1.7) always holds since

$$1 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1-q+2(1+r)\bar{y}}{1+(1+r)\bar{y}}.$$

Substitute $\bar{y} = \frac{q-1+\sqrt{(q-1)^2+4p(1+r)}}{2(1+r)}$. We get

$$\frac{1-q+(q-1+\sqrt{(q-1)^2+4p(1+r)})}{\frac{1+q+\sqrt{(q-1)^2+4p(1+r)}}{2}} = 2\frac{\sqrt{(q-1)^2+4p(1+r)}}{q+1+\sqrt{(q-1)^2+4p(1+r)}} > 0$$

Also inequality (5.1.8) holds for all values of the parameters p,q and r since

$$1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}} + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+q}{1+(1+r)\bar{y}}.$$

Substitute $\bar{y} = \frac{q-1+\sqrt{(q-1)^2+4p(1+r)}}{2(1+r)}$. We get

$$\frac{1+q}{\frac{1+q+\sqrt{(q-1)^2+4p(1+r)}}{2}} = 2\frac{1+q}{q+1+\sqrt{(q-1)^2+4p(1+r)}} > 0.$$

Inequality (5.1.9) is equivalent to

$$1 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}[\frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}] > 0. \tag{5.1.11}$$

Note that we take $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$ as a common factor. Now, add -1 to both sides of inequality (5.1.11), we have

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}[\frac{(1+r)\bar{y}-q}{1+\bar{y}+r\bar{y}}] > -1 \tag{5.1.12}$$

Multiply both sides of (5.1.12) by $\frac{1+\bar{y}+r\bar{y}}{(1+r)\bar{y}-q}$, for $(1+r)\bar{y}-q < 0$ we have

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < \frac{1+\bar{y}+r\bar{y}}{q-(1+r)\bar{y}}.$$

Inequality (5.1.10) is equivalent to

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}[\frac{-\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}] > -1$$

or

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}[\frac{-\bar{y}-q+r\bar{y}}{1+\bar{y}+r\bar{y}}] > -1. \tag{5.1.13}$$

Note that for $(1+r)\bar{y}-q<0$, $r\bar{y}-\bar{y}-q<0$. So if we multiply both sides of (5.1.13) by $\frac{1+\bar{y}+r\bar{y}}{r\bar{y}-\bar{y}-q}$, we have

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}.$$

Note that for $r\bar{y}-\bar{y}-q < r\bar{y}+\bar{y}-q < 0$,

$$0 < q-r\bar{y}-\bar{y} < q-r\bar{y}+\bar{y},$$

and hence

$$\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}-\bar{y}}.$$

So for $q-(1+r)\bar{y}>0$ if $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$ then $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}-\bar{y}}$, and hence for $q-(1+r)\bar{y}>0$ if inequality (5.1.10) holds, then inequality (5.1.9) holds.

Note that if $q-(1+r)\bar{y}>0$, we have

$$q-(1+r)(\frac{q-1+\sqrt{(q-1)^2+4p(1+r)}}{2(1+r)} > 0$$

or

$$q-\frac{q-1+\sqrt{(q-1)^2+4p(1+r)}}{2} > 0$$

$$q+1-\sqrt{(q-1)^2+4p(1+r)} > 0$$

$$q+1 > \sqrt{(q-1)^2+4p(1+r)}$$

take the square of both sides, we get

$$q^2+2q+1 > q^2-2q+1+4p(1+r)$$

or

$$4q > 4p(1+r)$$

$$p < \frac{q}{1+r}.$$

So for $p < \frac{q}{1+r}$ the zero solution is asymptotically stable if

$$\frac{q - r\bar{y}}{1 + \bar{y} + r\bar{y}} < \frac{1 + \bar{y} + r\bar{y}}{q - r\bar{y} + \bar{y}} \tag{5.1.14}$$

Note that if we fix q and r and choose p as a parameter where $p < \frac{q}{1+r}$, then the stability is exchange at the value of p that satisfies equation $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$. Name this value as p^*.

5.2 Bifurcation Of The Rational Difference Equation $y_{n+1} = \frac{p+qy_{n-2}}{1+y_n+ry_{n-2}}$

In this section we find the type of bifurcation that occurs at $p = p^*$ as p is the bifurcation parameter. Recall from previous chapters that equation (5.0.1) has no positive distinct periodic solutions of prime period two. We focus our attention on Neimark-Sacker bifurcation.

Theorem 5.1. *The characteristic polynomial (5.1.3) $p(\lambda)$ has two complex conjugate roots if one of the following cases holds*

1. $q - r\bar{y} < 0$
2. $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3$

Proof:

$$p(\lambda) = -\lambda^3 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$$

$$\acute{p}(\lambda) = -3\lambda^2 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda$$

$\acute{p}(\lambda) = 0$ at $\lambda_1^* = -\frac{2}{3}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})$ or $\lambda_2^* = 0$.
Since $\bar{y} > 0$, $\lambda_1^* < \lambda_2^*$. $p(\lambda)$ has local minimum value at $\lambda = \lambda_1^*$ and local maximum

value at $\lambda = \lambda_2^*$. Note that $\lim_{\lambda\to-\infty} p(\lambda) = \infty$ and $\lim_{\lambda\to\infty} p(\lambda) = -\infty$.
So $p(\lambda)$ has only one real root if one of the following cases holds

1. $p(\lambda_1^*) > 0$ and hence $p(\lambda_2^*) > p(\lambda_1^*) > 0$.

2. $p(\lambda_2^*) < 0$ and hence $p(\lambda_1^*) < p(\lambda_2^*) < 0$.

So $p(\lambda)$ has two conjugate complex roots if one of the following holds

1. $p(\lambda_1^*) = -\frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > 0$.

2. $p(\lambda_2^*) = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 0$.

Consider case one. Note that $p(0) = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > 0$ and $p(1) = -1 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}} + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$. Substitute the value of $\bar{y}$, we have $p(1) = -2\frac{\sqrt{(q-1)^2+4p(1+r)}}{q+1+\sqrt{(q-1)^2+4p(1+r)}} < 0$. So $p(\lambda)$ has a real root ξ such that $\xi \in (0,1)$.
In the second, case by similar argument we can show that $p(\lambda)$ has a real root of modulus less than one. Note that $p(0) < 0$ and $p(-1) > 0$ in this case.$\diamond$
Consider the case where $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3$. We will find where the conditions of Neimark-Sacker conditions hold.

Theorem 5.2. *For $p < \frac{q}{1+r}$, the characteristic polynomial $p(\lambda)$ has two complex conjugate roots of modulus one and a real root of modulus less than one at $p = p^*$ where p^* satisfies the equation $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$ if $q > 3$. Moreover, if $p^* >$ $\frac{\left(2(1+r)\left(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\right)-(q-1)\right)^2-(q-1)^2}{4(1+r)}$, then Neimark-Sacker conditions hold.*

To prove this theorem we need *Viète formula*.

Theorem 5.3. [1](*Viète formula*) *Given any polynomial of degree n, say*

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

with roots $r_1, r_2, \ldots, r_n$. $Vi\grave{e}teformula$ *say that*
$r_1 + r_2 + \ldots + r_n = -\frac{a_{n-1}}{a_n}$,
$(r_1r_2 + r_1r_3 + \ldots + r_1r_n) + (r_2r_3 + r_2r_4 + \ldots + r_2r_n) + \ldots + r_{n-1}r_n = \frac{a_{n-2}}{a_n}$,
$(r_1r_2r_3 + r_1r_2r_4 + \ldots + r_1r_2r_n) + (r_1r_3r_4 + r_1r_3r_5 + \ldots + r_1r_3r_n) + \ldots + r_{n-2}r_{n-1}r_n = -\frac{a_{n-3}}{a_n}$,
$\vdots$
$r_1r_2r_3 \ldots r_n = (-1)^n \frac{a_0}{a_n}$.

Proof of theorem (5.2): Consider that $q > 3$ and $p < \frac{q}{1+r}$. Note that for $p < \frac{q}{1+r}$, we have $q - (1+r)\bar{y} > 0$ and hence $q - r\bar{y} > \bar{y}$. Recall that $1 > (\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2$ so

$$1 > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2$$

and

$$\bar{y} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2\bar{y}$$

so

$$q - r\bar{y} > \bar{y} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2\bar{y}$$

multiply by $\frac{1}{1+\bar{y}+r\bar{y}}$, we get

$$\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} > \frac{4}{27}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3.$$

So in this case the characteristic polynomial has two complex conjugate roots and another real root of modulus less than one as we have shown in the proof of theorem (5.1). Now we will show that the modulus of the conjugate roots equals one.
Let λ_1, λ_2 and λ_3 be the roots of $p(\lambda)$ where λ_1 and λ_2 are the conjugate roots and λ_3 is the real root. Recall that $\lambda_3 = \xi$ has modulus less than one.
Apply $Vi\grave{e}te$ *theorem* to $p(\lambda)$

$$p(\lambda) = -\lambda^3 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$$

$a_0 = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$, $a_1 = 0$, $a_2 = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}}$ and $a_3 = -1$.

$$\lambda_1 + \lambda_2 + \lambda_3 = -\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \tag{5.2.1}$$

$$\lambda_1\lambda_2\lambda_3 = \frac{q - r\bar{y}}{1 + \bar{y} + r\bar{y}} \tag{5.2.2}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 0. \tag{5.2.3}$$

If λ_1 and λ_2 has modulus equal one then $\lambda_1\lambda_2 = 1$. From (5.2.2) we get

$$\lambda_3 = \frac{q - r\bar{y}}{1 + \bar{y} + r\bar{y}}.$$

Substitute λ_3 in equation (5.2.1), we get

$$\lambda_1 + \lambda_2 + \frac{q - r\bar{y}}{1 + \bar{y} + r\bar{y}} = -\frac{\bar{y}}{1 + \bar{y} + r\bar{y}}$$

$$\lambda_1 + \lambda_2 = -\frac{q - r\bar{y} + \bar{y}}{1 + \bar{y} + r\bar{y}}. \tag{5.2.4}$$

Also substitute λ_3 in equation (5.2.3), we get

$$\lambda_1 + \lambda_2 = -\frac{1}{\lambda_3} = -\frac{1 + \bar{y} + r\bar{y}}{q - r\bar{y}}.$$

That implies

$$\frac{q - r\bar{y}}{1 + \bar{y} + r\bar{y}} = \frac{1 + \bar{y} + r\bar{y}}{q - r\bar{y} + \bar{y}}.$$

This shows that at $p = p^*$ where p^* satisfies $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$, $p(\lambda)$ has two complex conjugate roots of modulus one and a real root of modulus less than one for $p < \frac{q}{1+r}$. As p is the bifurcation parameter and q and r are fixed, the bifurcation point is p^* which satisfies

$$\frac{q - r\bar{y}}{1 + \bar{y} + r\bar{y}} = \frac{1 + \bar{y} + r\bar{y}}{q - r\bar{y} + \bar{y}}$$

$$(1 + \bar{y} + r\bar{y})^2 = (q - r\bar{y} + \bar{y})(q - r\bar{y})$$

$$(1 + r)^2\bar{y}^2 + 2(1 + r)\bar{y} + 1 = q^2 - qr\bar{y} + q\bar{y} - qr\bar{y} + r^2\bar{y}^2 - r\bar{y}^2$$

$$(1 + 3r)\bar{y}^2 + (2(1 + r) + q(2r - 1))\bar{y} - (q^2 - 1) = 0. \tag{5.2.5}$$

Equation (5.2.5) is a quadratic equation has the following roots

$$\bar{y} = \frac{-(2(1 + r) + q(2r - 1)) \pm \sqrt{(2(1 + r) + q(2r - 1))^2 + 4(q^2 - 1)(1 + 3r)}}{2(1 + 3r)}.$$

Since $\bar{y} > 0$, for $q^2 > 1$

$$\bar{y} = \frac{-(2(1+r)+q(2r-1)) + \sqrt{(2(1+r)+q(2r-1))^2 + 4(q^2-1)(1+3r)}}{2(1+3r)}.$$

Substitute the value of $\bar{y}$, we have

$$\frac{q-1+\sqrt{(q-1)^2+4p^*(1+r)}}{2(1+r)} = \frac{-(2(1+r)+q(2r-1))+\sqrt{(2(1+r)+q(2r-1))^2+4(q^2-1)(1+3r)}}{2(1+3r)}$$

$$\sqrt{(q-1)^2+4p^*(1+r)} = 1-q+2(1+r)\left(\frac{-(2(1+r)+q(2r-1))+\sqrt{(2(1+r)+q(2r-1))^2+4(q^2-1)(1+3r)}}{2(1+3r)}\right)$$

$$p^* = \frac{\left(1-q+2[1+r][\frac{-(2(1+r)+q(2r-1))+\sqrt{(2(1+r)+q(2r-1))^2+4(q^2-1)(1+3r)}}{2(1+3r)}]\right)^2 - (q-1)^2}{4(1+r)}.$$

To check if Neimark-Saker bifurcation exists at p^* we must show that $e^{ik\theta^*} \neq 1$ for $k = 1,2,3,4$ and $\acute{r}(p^*) \neq 0$ where $\lambda_{1,2} = \cos\theta^* \pm i\sin\theta^*$.
To show that $e^{i\theta^*} \neq 1$, let $\lambda = \cos\theta + i\sin\theta$ and $\bar{\lambda} = \cos\theta - i\sin\theta$ be the complex roots of $p(\lambda)$ at p^*. Substitute λ in $p(\lambda)$, we have

$$-\lambda^3 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = 0$$

or

$$\lambda^3 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 - \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = 0. \tag{5.2.6}$$

Recall that at p^* $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$ so equation (5.2.6) becomes

$$\lambda^3 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = 0. \tag{5.2.7}$$

By similar argument substitute $\bar{\lambda}$ in $p(\lambda)$ we get

$$\bar{\lambda}^3 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2 - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = 0. \tag{5.2.8}$$

Multiply equation (5.2.7) by $\bar{\lambda}^2$ we have

$$\lambda + \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}\bar{\lambda}^2 = 0. \tag{5.2.9}$$

Also multiply equation (5.2.8) by λ^2, we have

$$\bar{\lambda} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}\lambda^2 = 0. \tag{5.2.10}$$

Add (5.2.9) to (5.2.10), we get

$$\lambda + \bar{\lambda} + 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}}) - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}(\lambda^2 + \bar{\lambda}^2) = 0 \tag{5.2.11}$$

Note that $\lambda + \bar{\lambda} = 2\cos\theta$ and $\lambda^2 + \bar{\lambda}^2 = 4\cos^2\theta - 2$.
Equation (5.2.11) becomes

$$2\cos\theta + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}} - \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}(4\cos^2\theta - 2) = 0$$

or

$$-4(\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}})\cos^2\theta + 2\cos\theta + 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}}) + 2(\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}) = 0. \tag{5.2.12}$$

From equation (5.2.4)

$$\lambda + \bar{\lambda} = -\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}$$

$$2\cos\theta = -\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}.$$

That implies

$$\cos\theta = -\frac{1}{2}(\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}).$$

Note that this is a root of equation (5.2.12) since

$$-4(\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}})(-\frac{1}{2}\times\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}})^2+2(-\frac{1}{2}\times\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}})+2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})+2(\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}})$$

$$= -2\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}} + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + 2\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$$

$$= 2(-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} + \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}) = 2(0) = 0$$

Note that $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 1$ or $q - r\bar{y} < 1 + \bar{y} + r\bar{y}$. To show this note that

$$0 < 4p(1+r)$$

add $(q-1)^2$ to the both sides, we get

$$(q-1)^2 < (q-1)^2 + 4p(1+r)$$

note, take the square root of the both sides

$$q-1 < \sqrt{(q-1)^2 + 4p(1+r)}$$

$$\frac{q-1}{2} < \frac{\sqrt{(q-1)^2 + 4p(1+r)}}{2}$$

or

$$q-1 < \frac{q-1+\sqrt{(q-1)^2 + 4p(1+r)}}{2}$$

or

$$q-1 < (1+r)\bar{y}$$

and hence,

$$q - r\bar{y} < 1 + \bar{y}$$

so

$$q - r\bar{y} < 1 + \bar{y} + r\bar{y}$$

and hence,

$$\frac{q - r\bar{y}}{1 + \bar{y} + r\bar{y}} < 1.$$

Since $\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 1$, $\cos\theta < -\frac{1}{2}$.

Also, note that $\frac{1}{2} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$. To show that we will use that for $q > 3$, we have $2 - q < 0$ and then

$$\frac{2-q}{1+r} < p$$

multiply both sides with 4, we have

$$8 - 4q < 4p(1+r)$$

add $(q-1)^2$ to the both sides, we get

$$q^2 - 6q + 9 < (q-1)^2 + 4p(1+r)$$

$$(q-3)^2 < (q-1)^2 + 4p(1+r)$$

take the square root of the both sides. Since we take $q > 3$, we get

$$q - 3 < \sqrt{(q-1)^2 + 4p(1+r)}$$

add $q-1$ to the both sides

$$2q - 4 < q - 1 + \sqrt{(q-1)^2 + 4p(1+r)}$$

$$q - 2 < (1+r)\bar{y}$$

and hence,

$$q - 2 < (1+3r)\bar{y}$$

or

$$q - r\bar{y} + \bar{y} < 2 + 2\bar{y} + 2r\bar{y}$$
$$\frac{1}{2} < \frac{1+\bar{y}+r\bar{y}}{q - r\bar{y} + \bar{y}}$$

Since $\frac{1}{2} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$, $\cos\theta > -1$.
So at p^* where $\frac{1}{2} < \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 1$, there exist $\theta_0 \in (\frac{\pi}{2}, \pi)$ such that $-1 < \cos\theta_0 = -\frac{1}{2}(\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}) < -\frac{1}{2}$. Note that $e^{ik\theta_0} \neq 1$ for $k = 1, 2, 3, 4$.

To check if $\acute{r}(p^*) \neq 0$, it is enough to show that $\frac{d|\lambda|^2}{dp} \mid_{p=p^*} \neq 0$.

$$p(\lambda) = -\lambda^3 - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2 + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}.$$

$$\frac{d\mid\lambda\mid^2}{dp}\mid_{p=p^*} = \frac{d(\lambda\bar{\lambda})}{dp}\mid_{p=p^*} = [\lambda\frac{d\bar{\lambda}}{dp} + \bar{\lambda}\frac{d\lambda}{dp}]\mid_{p=p^*}$$
$$= \lambda(\frac{dp(\bar{\lambda})}{dp}.\frac{d\bar{\lambda}}{dp(\bar{\lambda})}) + \bar{\lambda}(\frac{dp(\lambda)}{dp}.\frac{d\lambda}{dp(\lambda)}).$$

Note that

$$\frac{dp(\lambda)}{d\lambda} = -3\lambda^2 - 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda$$

To find $\frac{dp(\lambda)}{dp}$, note that

$$\frac{d}{dp}(\frac{\bar{y}}{1+\bar{y}+r\bar{y}}) = \frac{(1+\bar{y}+r\bar{y}) \times \frac{1}{2(1+r)} \times \frac{1}{2} \times \frac{1}{\sqrt{(q-1)^2+4p(1+r)}} \times 4(1+r)}{(1+\bar{y}+r\bar{y})^2}$$

$$-\frac{\bar{y}(1+r)\times\frac{1}{2(1+r)}\times\frac{1}{2}\times\frac{1}{\sqrt{(q-1)^2+4p(1+r)}}\times 4(1+r)}{(1+\bar{y}+r\bar{y})^2}$$

$$=\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}$$

Also

$$\frac{d}{dp}\left(\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}\right)=\frac{(1+\bar{y}+r\bar{y})\times -r\times\frac{1}{2(1+r)}\times\frac{1}{2\times\sqrt{(q-1)^2+4p(1+r)}}\times 4(1+r)}{(1+\bar{y}+r\bar{y})^2}$$

$$-\frac{(q-r\bar{y})\times(1+r)\times\frac{1}{2(1+r)}\frac{1}{2\times\sqrt{(q-1)^2+4p(1+r)}}\times 4(1+r)}{(1+\bar{y}+r\bar{y})^2}$$

$$=-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}.$$

Now,

$$\frac{d\mid\lambda\mid}{dp}=\bar{\lambda}\Bigg(\frac{-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\lambda^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}}{-3\lambda^2-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda}\Bigg)$$

$$+\lambda\Bigg(\frac{-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\bar{\lambda}^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}}{-3\bar{\lambda}^2-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}}\Bigg)$$

At p^* $\lambda\bar{\lambda}=1$, so we have

$$\frac{d\mid\lambda\mid}{dp}=\Bigg(\frac{-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\lambda^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}}{-3\lambda^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2}\Bigg)$$

$$+\Bigg(\frac{-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\bar{\lambda}^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}}{-3\bar{\lambda}^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2}\Bigg)$$

$$=\Bigg(\frac{\left(-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\lambda^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\right)(-3\bar{\lambda}^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2)}{(-3\lambda^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2)(-3\bar{\lambda}^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2)}\Bigg)$$

$$+\Bigg(\frac{\left(-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\bar{\lambda}^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\right)(-3\lambda^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2)}{(-3\lambda^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2)(-3\bar{\lambda}^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2)}\Bigg).$$

The denominator is non zero term since

$$(-3\lambda^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2)(-3\bar{\lambda}^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2) = 9+6\frac{\bar{y}}{1+\bar{y}+r\bar{y}}(\lambda+\bar{\lambda})+4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2.$$

At p^* $\lambda+\bar{\lambda} = -\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}}$, so the denominator becomes

$$9-6\frac{\bar{y}(q-r\bar{y}+\bar{y})}{(1+\bar{y}+r\bar{y})^2}+4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 = 9-6\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}-2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2$$

Note that $\frac{\bar{y}}{1+\bar{y}+r\bar{y}} < 1$ so $-(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 > -1$ and $-\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2} > -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}$.
So

$$9-6\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}-2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 > 9-6\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}-2$$

and since at p^* $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} < 1$,

$$9-6\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}-2 > 9-6-2 = 1 > 0.$$

It remains to show that the numerator is non zero term.
The numerator is

$$\Big(-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\lambda^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\Big)\Big(-3\bar{\lambda}^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\bar{\lambda}^2\Big)+\Big(-\frac{1}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\bar{\lambda}^2-\frac{r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}-\frac{(1+r)(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\Big)\Big(-3\lambda^3-2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\lambda^2\Big)$$

$$= \frac{3}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}(\lambda+\bar{\lambda})+\Big(\frac{3r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}+\frac{3(1+r)(q-ry)}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}\Big)(\lambda^3+\bar{\lambda}^3)+\Big(\frac{2r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}+\frac{2(1+r)(q-ry)y}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}\Big)(\lambda^2+\bar{\lambda}^2)+\frac{4\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}.$$

Recall that at p^* $\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} = \frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}}$. Also at p^*, $\lambda+\bar{\lambda} = 2\cos\theta_0$, $\lambda^2+\bar{\lambda}^2 = 4\cos^2\theta_0-2$ and $\lambda^3+\bar{\lambda}^3 = 8\cos^3\theta_0-6\cos\theta_0$ where $\cos\theta_0 = -\frac{1}{2}(\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}})$.
The numerator at p^* is

$$-\frac{3(q-r\bar{y}+\bar{y})}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}-\frac{3r(q-r\bar{y}+\bar{y})^3}{(1+\bar{y}+r\bar{y})^4\sqrt{(q-1)^2+4p(1+r)}}-\frac{3(1+r)(q-r\bar{y}+\bar{y})^2}{(1+\bar{y}+r\bar{y})^4\sqrt{(q-1)^2+4p(1+r)}}$$
$$+\frac{9r(q-r\bar{y}+\bar{y})}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}+\frac{9(1+r)}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}}+\frac{2r\bar{y}(q-r\bar{y}+\bar{y})^2}{(1+\bar{y}+r\bar{y})^4\sqrt{(q-1)^2+4p(1+r)}}$$

$+\frac{2(1+r)(q-r\bar{y}+\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^4\sqrt{(q-1)^2+4p(1+r)}} - \frac{4r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}} - \frac{4(1+r)\bar{y}}{(1+\bar{y}+r\bar{y})^2(q-r\bar{y}+\bar{y})\sqrt{(q-1)^2+4p(1+r)}} + \frac{4\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}.$

Note that $-1 < \cos\theta^* < -\frac{1}{2}$ which implies that $1 < \frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}} < 2$ and $-\frac{1}{q-r\bar{y}+\bar{y}} > -\frac{1}{1+\bar{y}+r\bar{y}}$.

The numerator is greater than

$-\frac{6}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}} - \frac{12(1+r)}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}} + \frac{9r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}} + \frac{9(1+r)}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}} + \frac{2r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}} + \frac{2(1+r)\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}} - \frac{4r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}} - \frac{4(1+r)\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}} + \frac{4\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}}$

$$= \frac{2(1-r)\bar{y}}{(1+\bar{y}+r\bar{y})^3\sqrt{(q-1)^2+4p(1+r)}} - \frac{6+3(1+r)+2r\bar{y}}{(1+\bar{y}+r\bar{y})^2\sqrt{(q-1)^2+4p(1+r)}} + \frac{9r}{(1+\bar{y}+r\bar{y})\sqrt{(q-1)^2+4p(1+r)}}. \quad (5.2.13)$$

Term (5.2.13) is positive if $2(1-r)\bar{y} - (6+3(1+r)+2r\bar{y})(1+\bar{y}+r\bar{y}) + 9r(1+\bar{y}+r\bar{y})^2 > 0$. That is equivalent to $(9r^3+16r^2+7r)\bar{y}^2 + (13r^2+16r-7)\bar{y} + 6r - 9 > 0$

or

$$\bar{y} > \frac{-(13r^2+16r-7) + \sqrt{(13r^2+16r-7)^2 + 4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}.$$

Substitute the value of $\bar{y}$, we get

$$\frac{q-1+\sqrt{(q-1)^2+4p^*(1+r)}}{2(1+r)} > \frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}$$

multiply the both sides by $2(1+r)$ and then add $-(q-1)$ for the both sides, we get

$$\sqrt{(q-1)^2 + 4p^*(1+r)} > 2(1+r)\left(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\right) - (q-1)$$

take the square of the both sides, we obtain

$$(q-1)^2 + 4p^*(1+r) >$$
$$\Big(2(1+r)(\tfrac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}) - (q-1)\Big)^2$$

add $-(q-1)^2$ for the both sides and then multiply by $\frac{1}{4(1+r)}$, we get

$$p^* > \frac{\Big(2(1+r)\Big(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\Big) - (q-1)\Big)^2 - (q-1)^2}{4(1+r)}$$

If term (5.2.13) is grater than zero, then $\frac{d|\lambda|^2}{dp}\mid_{p=p^*} > 0$ and then Neimark-Sacker bifurcation conditions are satisfied.

5.3 Direction Of Neimark-Sacker Bifurcation

System (5.1.2) can be written as

$$Y_{n+1} = JY_n + G(Y_n) \tag{5.3.1}$$

where $J = \begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & 0 & \frac{\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $Y_n = \begin{pmatrix} w_n \\ v_n \\ u_n \end{pmatrix}$

$G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(\parallel Y \parallel^3)$

$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ B_2(Y,Y) \\ B_3(Y,Y) \end{pmatrix}$ and $C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ C_2(Y,Y,Y) \\ C_3(Y,Y,Y) \end{pmatrix}$

$B_i(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 X_i(\xi)}{\partial \xi_j \partial \xi_k}\mid_{\xi=0} (x_j y_k)$ and $C_i(x,y,z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 X_i(\xi)}{\partial \xi_j \partial \xi_k}\mid_{\xi=0} (x_j y_k z_l)$

$B_1(\phi,\psi) = \frac{2(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}\phi_1\psi_1 - \frac{2r\bar{y}}{(1+\bar{y}+r\bar{y})^2}\phi_3\psi_3 + \frac{qr-(r^2+1)\bar{y}}{(1+\bar{y}+r\bar{y})^2}[\phi_3\psi_1 + \phi_1\psi_3],$

$B_2(\phi,\psi) = B_3(\phi,\psi) = 0,$

$C_1(\phi,\psi,\eta) = -\frac{6(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3}\phi_1\psi_1\eta_1 + \frac{2\bar{y}-4r(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^3}\Big(\phi_1\psi_1\eta_3+\phi_3\psi_1\eta_1+\phi_1+\psi_3\eta_1\Big) + \frac{2r^3\bar{y}+4r\bar{y}-2r^2q}{(1+\bar{y}+r\bar{y})^3}\Big(\phi_1\psi_3\eta_3+$
$\phi_3\psi_1\eta_3 + \phi_3\psi_3\eta_1\Big) + \frac{6r^2\bar{y}}{(1+\bar{y}+r\bar{y})^3}\phi_3\psi_3\eta_3,$

$C_2(\phi,\psi,\eta) = C_3(\phi,\psi,\eta) = 0.$
Recall that $\theta_0 = \cos^{-1}(-\frac{q-r\bar{y}+\bar{y}}{2(1+\bar{y}+r\bar{y})})$. Let q and p^* be the eigenvectors corresponding to the eigenvalues $\lambda = \cos\theta_0 + i\sin\theta_0 = e^{i\theta_0}$ and $\bar{\lambda} = \cos\theta_0 - i\sin\theta_0 = e^{-i\theta_0}$, respectively.
To find q we solve the equation

$$(J - e^{i\theta_0}I)q = 0 \tag{5.3.2}$$

or

$$\begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{i\theta_0} & 0 & \frac{\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 & -e^{i\theta_0} & 0 \\ 0 & 1 & -e^{i\theta_0} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $q \sim \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$. Let $q_2 = 1$, the second equation becomes

$$q_1 - e^{i\theta_0} = 0$$

which implies $q_1 = e^{i\theta_0}$ and the third equation implies

$$1 - e^{i\theta_0}q_3 = 0.$$

We get $q_3 = e^{-i\theta_0}$. We have $q \sim \begin{pmatrix} e^{i\theta_0} \\ 1 \\ e^{-i\theta_0} \end{pmatrix}$. Now, we must verify that this eigenvector q satisfies the first equation $(-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{i\theta_0})q_1 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}q_3 = 0$ or

$$(-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{i\theta_0})e^{i\theta_0} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-i\theta_0} = 0.$$

Note that equation (5.3.2) has a nonzero solution if the matrix $J - e^{i\theta_0}I$ is singular matrix which means $\mid J - e^{i\theta_0}I \mid = 0$

$$\mid J - e^{i\theta_0}I \mid = (-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{i\theta_0})e^{2i\theta_0} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}} = 0. \tag{5.3.3}$$

Equation (5.3.3) is equivalent to

$$e^{i\theta_0}\Big((-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{i\theta_0})e^{i\theta_0} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-i\theta_0}\Big) = 0$$

which implies

$$(-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{i\theta_0})e^{i\theta_0} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-i\theta_0} = 0$$

So q satisfies the first equation.

To find p^*, we solve the equation $(J - e^{-i\theta_0}I)^T p^* = 0$ or

$$\begin{pmatrix} -\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{-i\theta_0} & 1 & 0 \\ 0 & -e^{-i\theta_0} & 1 \\ \frac{\bar{y}}{1+\bar{y}+r\bar{y}} & 0 & -e^{-i\theta_0} \end{pmatrix} \begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $p^* \sim \begin{pmatrix} p_1^* \\ p_2^* \\ p_3^* \end{pmatrix}$. Let $p_3^* = 1$. From the second equation we get

$$-e^{-i\theta_0}p_2^* + 1 = 0$$

which implies $p_2^* = e^{i\theta_0}$ and from the third equation we get

$$\frac{\bar{y}}{1+\bar{y}+r\bar{y}}p_1^* - e^{-i\theta_0} = 0.$$

We get $p_1^* = \frac{1+\bar{y}+r\bar{y}}{\bar{y}}e^{-i\theta_0}$. To show that this choice of $p^* \sim \begin{pmatrix} \frac{1+\bar{y}+r\bar{y}}{\bar{y}}e^{-i\theta_0} \\ e^{i\theta_0} \\ 1 \end{pmatrix}$ satisfies the first equation, note that $\mid (J - e^{-i\theta_0}I)^T \mid = 0$ implies

$$(-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{-i\theta_0})e^{-2i\theta_0} + \frac{\bar{y}}{1+\bar{y}+r\bar{y}} = 0.$$

The previous equation is equivalent to

$$\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-i\theta_0}\Big((-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{-i\theta_0})(\frac{1+\bar{y}+r\bar{y}}{\bar{y}}e^{-i\theta_0}) + e^{i\theta_0}\Big) = 0$$

which implies

$$(-\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - e^{-i\theta_0})(\frac{1+\bar{y}+r\bar{y}}{\bar{y}}e^{-i\theta_0}) + e^{i\theta_0} = 0.$$

So p^* satisfies the first equation.
To normalize q and p^*, we must find ζ such that $< \zeta p^*, q >= 1$, where $< .,. >$ is the standard scalar product in $\mathbb{C}^3$.

$$< \zeta p^*, q >= \zeta \sum_{i=1}^{3} \bar{p_i^*} q_i = \zeta \frac{1+\bar{y}+r\bar{y}}{\bar{y}} e^{-i\theta_0} + 2e^{-i\theta_0}$$

Tack $\zeta = \frac{1}{\frac{1+\bar{y}+r\bar{y}}{\bar{y}} e^{2i\theta_0} + 2e^{-i\theta_0}}$. So take $p = \zeta * p^*$. We have $< p, q >= 1$.
The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is two-dimensional and is spanned by $\{Re(q), Im(q)\}$. The real eigenspace T^s corresponding to the real eigenvalues of J is one-dimensional. Any vector $x \in \mathbb{R}^3$ can be decomposed as

$$x = zq + \bar{z}\bar{q} + y$$

where $z \in \mathbb{C}^1$, $\bar{z}\bar{q} \in T^c$ and $y \in T^s$. The complex variable z is a coordinate on T^c. We have

$$z =< p, x >,$$

$$y = x- < p, x > q- < \bar{p}, x > \bar{q}.$$

In these coordinates, the map (5.3.1) takes the form

$$\tilde{z} = e^{i\theta_0} z+ < p, G(zq + \bar{z}\bar{q} + y) >,$$

$$\tilde{y} = Jy + G(zq + \bar{z}\bar{q} + y)- < p, G(zq + \bar{z}\bar{q} + y) > q- < \bar{p}, G(zq + \bar{z}\bar{q} + y) > \bar{q}.$$

The previous system can be written as

$$\tilde{z} = e^{i\theta_0} z + \frac{1}{2} G_{20} z^2 + G_{11} z\bar{z} + \frac{1}{2} G_{02} \bar{z}^2 + \frac{1}{2} G_{21} z^2 \bar{z}+ < G_{10}, y > z+ < G_{01}, y > \bar{z},$$

$$\tilde{y} = Jy + \frac{1}{2} H_{20} z^2 + H_{11} z\bar{z} + \frac{1}{2} H_{02} \bar{z}^2 + \frac{1}{2} H_{21} z^2 \bar{z}$$

where

$$G_{20} =< p, B(q, q) >, G_{11} =< p, B(q, \bar{q} >, G_{02} =< p, B(\bar{q}, \bar{q}) >, G_{21} =< p, C(q, q, \bar{q}) >$$

and

$$H_{20} = B(q, q)- < p, B(q, q) > q- < \bar{p}, B(q, q) > \bar{q}, H_{11} = B(q, \bar{q})- < p, B(q, \bar{q}) > q- < \bar{p}, B(q, \bar{q}) >$$

and

$$< G_{10}, y >=< p, B(q, y) >, < G_{01}, y >=< p, B(\bar{q}, y) >$$

where the scaler product is in $\mathbb{C}^3$.
From the center manifold theorem , there exists a center manifold W^c which can be approximated as

$$Y = V(z, \bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2$$

where $< q, w_{ij} >= 0$. The vectors $w_{ij} \in \mathbb{C}^3$ can be found from the linear equations

$$(e^{2i\theta_0}I_3 - J)w_{20} = H_{20},$$

$$(I_3 - J)w_{11} = H_{11},$$

$$(e^{-2i\theta_0}I_3 - J)w_{02} = H_{02}.$$

These equations has unique solutions. Note that the matrices (I_3-J) and $(e^{\pm 2i\theta_0}I_3 - J)$ are invertible in $\mathbb{C}^3$ since 1 and $e^{\pm 2i\theta_0}$ are not eigenvalues of J. Recall that $e^{i\theta_0} \neq 1$.
So z can be written as
$\tilde{z} = e^{i\theta_0}\bar{z} + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}[G_{21} + 2 < p, B(q, (I-J)^{-1}H_{11}) > + < p, B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}H_{20}) >]z^2\bar{z} + \ldots.$
Taking into account the identities

$$(I-J)^{-1}q = \frac{1}{1-e^{i\theta_0}}q,\ (e^{2i\theta_0}I - J)^{-1}q = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}q,\ (i-j)^{-1}\bar{q} = \frac{1}{1-e^{i\theta_0}}\bar{q}$$

and

$$(e^{2i\theta_0}I - J)^{-1}\bar{q} = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}\bar{q}.$$

Also z can be written using the map

$$\tilde{z} = e^{i\theta_0}z + \sum_{k,l\geq 2}\frac{1}{k!j!}g_{kj}z^k\bar{z}^j \tag{5.3.4}$$

where $g_{20} =< p, B(q, q) >,\ g_{11} =< p, B(q, \bar{q}) >,\ g_{02} =< p, B(\bar{q}, \bar{q}) >$
and $g_{21} =< p, C(q, q, \bar{q}) > +2 < p, B(q, (I-J)^{-1}B(q, \bar{q})) > + < p, B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}B(q, q)) > + \frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{i\theta_0}} < p, B(q, q) >< p, B(q, \bar{q}) > - \frac{2}{1-e^{-i\theta_0}} |< p, B(q, \bar{q}) >|^2$

$-\frac{e^{i\theta_0}}{e^{3i\theta_0}-1} \mid < p, B(\bar{q},\bar{q}) > \mid^2$.

The map (5.3.4) can be transformed into the form

$$\tilde{z} = e^{i\theta_0} z(1 + d(p^*)) \mid z^2 \mid$$

where p^* is the value of the bifurcation parameter p where the Neimark-Sacker bifurcation exists and the real number $a(p^*) = Re(d(p^*))$, that determines the direction of bifurcation of the closed invariant curve, can be computed by the following formula

$$a(p^*) = Re\Big(\frac{e^{-i\theta_0} g_{21}}{2}\Big) - Re\Big(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})} g_{20} g_{11}\Big) - \frac{1}{2} \mid g_{11} \mid^2 - \frac{1}{4} \mid g_{02} \mid^2 .$$

Now, we compute $a(p^*)$. Recall that $g_{20} =< p, B(q,q) >$.

$$\text{Where } B(q,q) = \left(\begin{array}{c} \frac{2\left(q-r\bar{y}\right)e^{2i\theta_0}-2r\bar{y}e^{-2i\theta_0}+2qr-2\left(r^2+1\right)\bar{y}}{(1+\bar{y}+r\bar{y})^2} \\ 0 \\ 0 \end{array}\right).$$

$$g_{20} = \zeta \frac{1+\bar{y}+r\bar{y}}{\bar{y}} e^{-i\theta_0} \Big(\frac{2qe^{2i\theta_0} + 2qr - 2(r^2+1)\bar{y} - 4r\bar{y}\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2}\Big)$$

or

$$g_{20} = \frac{1}{e^{3i\theta_0} + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}} \Big(\frac{2qe^{2i\theta_0} + 2qr - 2(r^2+1)\bar{y} - 4r\bar{y}\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2}\Big)$$

$$g_{11} =< p, B(q,\bar{q}) >, \text{ where } B(q,\bar{q}) = \left(\begin{array}{c} \frac{2\left(q-r\bar{y}\right)-2r\bar{y}+2\left(qr-(r^2+1)\bar{y}\right)\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2} \\ 0 \\ 0 \end{array}\right).$$

So

$$g_{11} = \zeta \frac{1+\bar{y}+r\bar{y}}{\bar{y}} e^{-i\theta_0} \Big(\frac{2(q-r\bar{y}) - 2r\bar{y} + 2(qr-(r^2+1)\bar{y})\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2}\Big)$$

or

$$g_{11} = \frac{1}{e^{3i\theta_0} + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}} \Big(\frac{2(q-r\bar{y}) - 2r\bar{y} + 2(qr-(r^2+1)\bar{y})\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2}\Big)$$

$$g_{02} =< p, B(\bar{q},\bar{q}) > \text{ where } B(\bar{q},\bar{q}) = \begin{pmatrix} \frac{2qe^{-2i\theta_0}-4r\bar{y}\cos 2\theta+2\left(qr-(r^2+1)\right)}{(1+\bar{y}+r\bar{y})^2} \\ 0 \\ 0 \end{pmatrix}.$$

So

$$g_{02} = \zeta\frac{1+\bar{y}+r\bar{y}}{\bar{y}}e^{-i\theta_0}\Big(\frac{2qe^{-2i\theta_0}-4r\bar{y}\cos 2\theta+2(qr-(r^2+1))}{(1+\bar{y}+r\bar{y})^2}\Big)$$

or

$$g_{02} = \frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\Big(\frac{2qe^{-2i\theta_0}-4r\bar{y}\cos 2\theta+2(qr-(r^2+1))}{(1+\bar{y}+r\bar{y})^2}\Big).$$

$g_{21} =< p, C(q,q,\bar{q}) > +2 < p, B(q,(I-J)^{-1}B(q,\bar{q})) > + < p, B(\bar{q},(e^{2i\theta_0}I - J)^{-1}B(q,q)) > + \frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{i\theta_0}} < p, B(q,q) >< p, B(q,\bar{q}) > - \frac{2}{1-e^{-i\theta_0}} |< p, B(q,\bar{q}) >|^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0}-1} |< p, B(\bar{q},\bar{q}) >|^2$.

$$C(q,q,\bar{q}) = \begin{pmatrix} \frac{(-6(q-r\bar{y})-4r^2(q-r\bar{y})+8r\bar{y})e^{i\theta_0}+(4\bar{y}-8r(q-r\bar{y})+6r^2\bar{y})e^{-i\theta_0}+(2\bar{y}-4r(q-r\bar{y}))e^{3i\theta_0}+(4r\bar{y}-2r^2(q-r\bar{y}))e^{-i\theta_0}}{(1+\bar{y}+r\bar{y})^3} \\ 0 \\ 0 \end{pmatrix}$$

$$< p, C(q,q,\bar{q}) >= \frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\Bigg(\frac{\Big(-6(q-r\bar{y})-4r^2(q-r\bar{y})+8r\bar{y}\Big)e^{i\theta_0}}{(1+\bar{y}+r\bar{y})^3}$$
$$+\frac{\Big(4\bar{y}-8r(q-r\bar{y})+6r^2\bar{y}\Big)e^{-i\theta_0}+\Big(2\bar{y}-4r(q-r\bar{y})\Big)e^{3i\theta_0}+\Big(4r\bar{y}-2r^2(q-r\bar{y})\Big)e^{-i\theta_0}}{(1+\bar{y}+r\bar{y})^3}\Bigg).$$

The second term in g_{21} is $< p, B(q,(I-J)^{-1}B(q,\bar{q})) >$

$$(I-J)^{-1} = \begin{pmatrix} \frac{1+q+\bar{y}}{1+\bar{y}+r\bar{y}} & 0 & -\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1+\bar{y}+r\bar{y}}{q+1} & \frac{\bar{y}}{q+1} & \frac{\bar{y}}{q+1} \\ \frac{1+\bar{y}+r\bar{y}}{q+1} & \frac{1+q+\bar{y}}{q+1} & \frac{\bar{y}}{q+1} \\ \frac{1+\bar{y}+r\bar{y}}{q+1} & \frac{1+q+\bar{y}}{q+1} & \frac{1+q+\bar{y}}{q+1} \end{pmatrix}$$

$$(I-J)^{-1}B(q,\bar{q}) = \begin{pmatrix} \frac{1}{q+1}\left(\frac{2(q-r\bar{y})-2r\bar{y}-2(qr-(r^2+1)\bar{y})\cos 2\theta_0}{1+\bar{y}+r\bar{y}}\right) \\ \frac{1}{q+1}\left(\frac{2(q-r\bar{y})-2r\bar{y}-2(qr-(r^2+1)\bar{y})\cos 2\theta_0}{1+\bar{y}+r\bar{y}}\right) \\ \frac{1}{q+1}\left(\frac{2(q-r\bar{y})-2r\bar{y}-2(qr-(r^2+1)\bar{y})\cos 2\theta_0}{1+\bar{y}+r\bar{y}}\right) \end{pmatrix} = \begin{pmatrix} S \\ S \\ S \end{pmatrix}$$

$$B(q,(I-J)^{-1}B(q,\bar{q})) = \begin{pmatrix} \frac{2\left(q-r\bar{y}\right)Se^{i\theta}-2r\bar{y}Se^{-i\theta}+2\left(qr-(r^2+1)\bar{y}\right)S\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2} \\ 0 \\ 0 \end{pmatrix}$$

$$< p, B(q,(I-J)^{-1}B(q,\bar{q})) >= \frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\left(\frac{2(q-r\bar{y})Se^{i\theta}-2r\bar{y}Se^{-i\theta}}{(1+\bar{y}+r\bar{y})^2}\right.$$
$$\left.+\frac{2\left(qr-(r^2+1)\bar{y}\right)S\cos 2\theta_0}{(1+\bar{y}+r\bar{y})^2}\right)$$

$$(e^{2i\theta_0}I-J)^{-1} = \begin{pmatrix} e^{2i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & 0 & -\frac{\bar{y}}{1+\bar{y}+r\bar{y}} \\ -1 & e^{2i\theta_0} & 0 \\ 0 & -1 & e^{2i\theta_0} \end{pmatrix}^{-1}$$

$$= \frac{1}{D}\begin{pmatrix} e^{4i\theta_0} & \frac{\bar{y}}{1+\bar{y}+r\bar{y}} & \frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{2i\theta_0} \\ e^{2i\theta_0} & e^{4i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{2i\theta_0} & \frac{\bar{y}}{1+\bar{y}+r\bar{y}} \\ 1 & e^{2i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} & e^{4i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{2i\theta_0} \end{pmatrix}$$

where D is the determinant of the matrix $(e^{2i\theta_0}I-J)$ such that $D = e^{4i\theta_0}(e^{2i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}) - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}$.

$$(e^{2i\theta_0}I-J)^{-1}B(q,q) = \begin{pmatrix} \frac{L}{D}e^{4i\theta_0} \\ \frac{L}{D}e^{2i\theta_0} \\ \frac{L}{D} \end{pmatrix}$$

where $L = \frac{2\left(q-r\bar{y}\right)e^{2i\theta_0}-2r\bar{y}e^{-2i\theta_0}+2\left(qr-(r^2+1)\bar{y}\right)}{(1+\bar{y}+r\bar{y})^2}$.

$$B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q) = \begin{pmatrix} \frac{L}{D}\left(\frac{2(q-r\bar{y})e^{3i\theta_0}-2r\bar{y}e^{i\theta_0}+(qr-(r^2+1)\bar{y})(e^{5i\theta_0}+e^{-i\theta_0})}{(1+\bar{y}+r\bar{y})^2}\right) \\ 0 \\ 0 \end{pmatrix}.$$

$$< p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q)) >= \frac{L}{D}\left(\frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}}\right)\left(\frac{2(q-r\bar{y})e^{3i\theta_0}-2r\bar{y}e^{i\theta_0}+(qr-(r^2+1)\bar{y})(e^{5i\theta_0}+e^{-i\theta_0})}{(1+\bar{y}+r\bar{y})^2}\right).$$

$a(p^*) = Re(\frac{e^{-i\theta_0}}{2} < p, c(q,q,\bar{q}) >) + Re(e^{-i\theta_0} < p, B(q,(I-J)^{-1}B(q,\bar{q})) >) + Re(\frac{e^{-i\theta_0}}{2} < p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q))) >)$. Now, we find the first term $N_1 = e^{-i\theta_0} < p, C(q,q,\bar{q}) >$
$N_1 = (\frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}})(\frac{-6(q-r\bar{y})-4r^2(q-r\bar{y})+8r\bar{y}}{(1+\bar{y}+r\bar{y})^3} + \frac{(4r\bar{y}-2r^2(q-r\bar{y}))e^{-4i\theta_0}}{(1+\bar{y}+r\bar{y})^3} + \frac{(4\bar{y}-8r(q-r\bar{y})+6r^2\bar{y})e^{-2i\theta_0}}{(1+\bar{y}+r\bar{y})^3}$
$+\frac{(2\bar{y}-4r(q-r\bar{y}))e^{2i\theta_0}}{(1+\bar{y}+r\bar{y})^3})$.

Multiply the numerator of N_1 by the conjugate of the denominator, the numerator becomes
$(4r\bar{y}-2r^2(q-r\bar{y}))e^{-7i\theta_0}+(4\bar{y}-8r(q-r\bar{y})+6r^2\bar{y})e^{-5i\theta_0}+(4r\bar{y}-2r^2(q-r\bar{y}))\frac{2\bar{y}}{1+\bar{y}+r\bar{y}}e^{-4i\theta_0}+(-6(q-r\bar{y})-4r^2(q-r\bar{y})+8r\bar{y})e^{-3i\theta_0}+(4\bar{y}-8r(q-r\bar{y})+6r^2\bar{y})\frac{2\bar{y}}{1+\bar{y}+r\bar{y}}e^{-2i\theta_0}+(2\bar{y}-4r(q-r\bar{y}))e^{-i\theta_0}+(-6(q-r\bar{y})-4r^2(q-r\bar{y})+8r\bar{y})\frac{2\bar{y}}{1+\bar{y}+r\bar{y}}+(2\bar{y}-4r(q-r\bar{y}))\frac{2\bar{y}}{1+\bar{y}+r\bar{y}}e^{2i\theta_0}$.

Let F_1 denotes the real part of the previous term
$F_1 = (4r\bar{y}-2r^2(q-r\bar{y}))\cos 7\theta_0 + (4\bar{y}-8r(q-r\bar{y})+6r^2\bar{y})\cos 5\theta_0 + (4r\bar{y}-2r^2(q-r\bar{y}))\frac{2\bar{y}}{1+\bar{y}+r\bar{y}}\cos 4\theta_0 + (-6(q-r\bar{y})-4r^2(q-r\bar{y})+8r\bar{y})\cos 3\theta_0 + [(2\bar{y}-4r(q-r\bar{y}))+(4\bar{y}-8r(q-r\bar{y})+6r^2\bar{y})]\frac{2\bar{y}}{1+\bar{y}+r\bar{y}}\cos 2\theta_0 + (2\bar{y}-4r(q-r\bar{y}))\cos\theta_0 + (-6(q-r\bar{y})-4r^2(q-r\bar{y})+8r\bar{y})\frac{2\bar{y}}{1+\bar{y}+r\bar{y}}$

Multiply the denominator of N_1 by it's conjugate, we have $F_2 = 1+4\frac{\bar{y}}{1+\bar{y}+r\bar{y}}\cos 3\theta_0 + 4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2$.

So $Re(N_1) = \frac{F_1}{(1+\bar{y}+r\bar{y})^3F_2}$.

Let $N_2 = e^{-i\theta_0} < p, B(q,(I-J)^{-1}B(q,\bar{q})) >= e^{-i\theta_0}(\frac{2(q-r\bar{y})Se^{i\theta_0}-2r\bar{y}Se^{-i\theta_0}+2(qr-(r^2+1)\bar{y})S\cos\theta_0}{(1+\bar{y}+r\bar{y})^2(e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}})})$

$= \frac{2(q-r\bar{y})S-2r\bar{y}Se^{-2i\theta_0}+2(qr-(r^2+1)\bar{y})Se^{-i\theta_0}\cos\theta_0}{(1+\bar{y}+r\bar{y})^2(e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}})}$.

Multiply the numerator of N_2 by the conjugate of the denominator, we get $-2r\bar{y}e^{-5i\theta_0} + 2(qr - (r^2+1)\bar{y})Se^{-4i\theta_0}\cos\theta_0 + 2(q - r\bar{y})Se^{-3i\theta_0} - \frac{4r\bar{y}^2S}{1+\bar{y}+r\bar{y}}e^{-2i\theta_0} + \frac{4(qr-(r^2+1)\bar{y})\bar{y}S}{1+\bar{y}+r\bar{y}}e^{-i\theta_0}\cos\theta_0 + \frac{4(q-r\bar{y})\bar{y}S}{1+\bar{y}+r\bar{y}}$.

Let F_3 be the real part of the previous term
$F_3 = -2r\bar{y}\cos 5\theta_0 + 2(qr-(r^2+1)\bar{y})S\cos 4\theta_0\cos\theta_0 + 2(q-r\bar{y})S\cos 3\theta_0 - \frac{4r\bar{y}^2S}{1+\bar{y}+r\bar{y}}\cos 2\theta_0 + \frac{4(qr-(r^2+1)\bar{y})\bar{y}S}{1+\bar{y}+r\bar{y}}\cos^2\theta_0 + \frac{4(q-r\bar{y})\bar{y}S}{1+\bar{y}+r\bar{y}}$.

Recall that $S = \frac{2(q-r\bar{y})-2r\bar{y}+2[qr-(r^2+1)\bar{y}]\cos 2\theta_0}{1+\bar{y}+r\bar{y}}$. So $Re(N_2) = \frac{F_3}{(1+\bar{y}+r\bar{y})^2F_2}$

Let $N_3 = e^{-i\theta_0} < p, B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}B(q,q)) >$
$= e^{-i\theta_0}[(\frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}})(\frac{L}{(1+\bar{y}+r\bar{y})^2D})(2(q - r\bar{y})e^{3i\theta_0} - 2r\bar{y}e^{i\theta_0} + (qr - (r^2+1)\bar{y})(e^{5i\theta_0} + e^{-i\theta_0}))]$

$= (\frac{1}{e^{3i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}})(\frac{L}{(1+\bar{y}+r\bar{y})^2D})(2(q - r\bar{y})e^{2i\theta_0} - 2r\bar{y} + (qr - (r^2+1)\bar{y})(e^{4i\theta_0} + e^{-2i\theta_0}))$.

Recall that $L = \frac{2(q-r\bar{y})e^{2i\theta_0}-2r\bar{y}e^{-2i\theta_0}+2(qr-(r^2+1)\bar{y})}{(1+\bar{y}+r\bar{y})^2}$ and $D = e^{6i\theta_0} + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{4i\theta_0} - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}$.

Substitute the value of L and D in N_3, the numerator is
$4(q-r\bar{y})^2e^{4i\theta_0} - 4r\bar{y}(q-r\bar{y}) + 4(q-r\bar{y})(qr-(r^2+1)\bar{y}-r\bar{y})e^{2i\theta_0} + 4r^2\bar{y}^2e^{-2i\theta_0} - 4r\bar{y}(qr - (r^2+1)\bar{y}) + 2(q - r\bar{y})(qr - (r^2+1)\bar{y})e^{6i\theta_0} - 2r\bar{y}(qr - (r^2+1)\bar{y})e^{2i\theta_0} + 2(qr - (r^2+1)\bar{y})^2e^{4i\theta_0} + 2(q-r\bar{y})(qr-(r^2+1)\bar{y}) - 2r\bar{y}(qr-(r^2+1)\bar{y})e^{-4i\theta_0} + 2(qr-(r^2+1)\bar{y})^2e^{-2i\theta_0}$.

Let $a_0 = -4r\bar{y}(q - r\bar{y}) - 4r\bar{y}(qr - (r^2+1)\bar{y}) + 2(q - r\bar{y})(qr - (r^2+1)\bar{y})$,
$a_1 - 2r\bar{y}(qr - (r^2+1)\bar{y})$,
$a_2 = 4r^2\bar{y}^2 + 2(qr - (r^2+1)\bar{y})^2$,
$a_3 = 4q(qr - (r^2+1)\bar{y}) - 6r\bar{y}(qr - (r^2+1)\bar{y}) - 4r\bar{y}(q - r\bar{y}))$,
$a_4 = 4(q - r\bar{y})^2 + 2(qr - (r^2+1)\bar{y})^2$,
$a_5 = 2(q - r\bar{y})(qr - (r^2+1)\bar{y})$.

The numerator is $a_5e^{6i\theta_0} + a_4e^{4i\theta_0} + a_3e^{2i\theta_0} + a_0 + a_2e^{-2i\theta_0} + a_1e^{-4i\theta_0}$

and the denominator is $(1+\bar{y}+r\bar{y})^4[e^{9i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{7i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{6i\theta_0}+2\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}e^{4i\theta_0}-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{3i\theta_0} - 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2]$.

The conjugate of the denominator is
$(1+\bar{y}+r\bar{y})^4[e^{-9i\theta_0} + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{-7i\theta_0} + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-6i\theta_0} + 2\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}e^{-4i\theta_0} - \frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-3i\theta_0} - 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2]$.
Multiply the numerator of N_3 by the conjugate of the denominator

$$[4(q-r\bar{y})^2e^{4i\theta_0} - 4r\bar{y}(q-r\bar{y}) + 4(q-r\bar{y})(qr-(r^2+1)\bar{y}-r\bar{y})e^{2i\theta_0} + 4r^2\bar{y}^2e^{-2i\theta_0}$$

$$-4r\bar{y}(qr-(r^2+1)\bar{y}) + 2(q-r\bar{y})(qr-(r^2+1)\bar{y})e^{6i\theta_0} - 2r\bar{y}(qr-(r^2+1)\bar{y})e^{2i\theta_0}$$

$$+2(qr-(r^2+1)\bar{y})^2e^{4i\theta_0} + 2(q-r\bar{y})(qr-(r^2+1)\bar{y}) - 2r\bar{y}(qr-(r^2+1)\bar{y})e^{-4i\theta_0}$$

$$+2(qr-(r^2+1)\bar{y})^2e^{-2i\theta_0}][1+\bar{y}+r\bar{y}]^4[e^{-9i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{-7i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-6i\theta_0}+2\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}e^{-4i\theta_0}$$

$$-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-3i\theta_0} - 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2]$$

$$= (1+\bar{y}+r\bar{y})^4[-2a_5(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2e^{6i\theta_0} - 2a_4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2e^{4i\theta_0} - a_5\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{3i\theta_0}$$

$$+(-2a_3(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2a_5\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2})e^{2i\theta_0} - a_4\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{i\theta_0}$$

$$+(-2a_0(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2a_4\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} + 2a_5\frac{\bar{y}}{1+\bar{y}+r\bar{y}}) + (-a_3\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + a_5\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})e^{-i\theta_0}$$

$$+(-2a_2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2+2a_3\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2}+2a_4\frac{\bar{y}}{1+\bar{y}+r\bar{y}})e^{-2i\theta_0}+(-a_0\frac{\bar{y}}{1+\bar{y}+r\bar{y}}+a_4\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}+a_5)e^{-3i\theta_0}$$

$$+(2a_0\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} - 2a_1(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2a_3\frac{\bar{y}}{1+\bar{y}+r\bar{y}})e^{-4i\theta_0} + (-a_2\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + a_3\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} + a_4)e^{-5i\theta_0}$$

$$+(2a_0\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + 2a_2\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2})e^{-6i\theta_0} + (a_0\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - a_1\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + a_3)e^{-7i\theta_0}$$

$$+(2a_1\tfrac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} + 2a_2\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})e^{-8i\theta_0} + (a_0 + a_2\tfrac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})e^{-9i\theta_0} + 2a_1\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-10i\theta_0}$$

$$+(a_1\tfrac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} + a_2)e^{-11i\theta_0} + a_1e^{-13i\theta_0}].$$

The real part of the previous term is $(1+\bar{y}+r\bar{y})^4F_4$ where F_4 is
$F_4 = (-2a_0(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2a_4\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} + 2a_5\frac{\bar{y}}{1+\bar{y}+r\bar{y}})$

$$+(-a_4\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}} - a_3\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}} + a_5\tfrac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})\cos\theta_0$$

$$+(-2a_2(\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2a_3\tfrac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} + 2a_4\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}} - 2a_3(\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2a_5\tfrac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2})\cos 2\theta_0$$

$$+(-a_5\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}} - a_0\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}} + a_4\tfrac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} + a_5)\cos 3\theta_0$$

$$+(2a_0\tfrac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} - 2a_1(\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2a_3\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}} - 2a_4(\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})^2)\cos 4\theta_0$$

$$+(-a_2\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}}+a_3\tfrac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}+a_4)\cos 5\theta_0+(2a_0\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}}+2a_2\tfrac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2}-2a_5(\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})^2)\cos 6\theta_0$$

$$(a_0\tfrac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} - a_1\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}} + a_3)\cos 7\theta_0 + (2a_1\tfrac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} + 2a_2\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})\cos 8\theta_0$$

$$+(a_0 + a_2\tfrac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})\cos 9\theta_0 + 2a_1\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}}\cos 10\theta_0 + (a_1\tfrac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} + a_2)\cos 11\theta_0 + a_1\cos 13\theta_0.$$

Also multiply the denominator of N_3 by it's conjugate
$(1+\bar{y}+r\bar{y})^4[e^{9i\theta_0}+\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{7i\theta_0}+2\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{6i\theta_0}+2\frac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}e^{4i\theta_0}-\frac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{3i\theta_0}-2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2](1$

$$+\bar{y}+r\bar{y})^4[e^{-9i\theta_0} + \tfrac{q-r\bar{y}}{1+\bar{y}+r\bar{y}}e^{-7i\theta_0} + 2\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-6i\theta_0} + 2\tfrac{\bar{y}(q-r\bar{y})}{(1+\bar{y}+r\bar{y})^2}e^{-4i\theta_0} - \tfrac{\bar{y}}{1+\bar{y}+r\bar{y}}e^{-3i\theta_0}$$

$$-2(\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})^2]$$

$$= (1+\bar{y}+r\bar{y})^8[-2(\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})^2e^{9i\theta_0} - 2(\tfrac{(q-r\bar{y})\bar{y}^2}{1+\bar{y}+r\bar{y}})^3e^{7i\theta_0} - (4(\tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})^3 + \tfrac{\bar{y}}{1+\bar{y}+r\bar{y}})e^{6i\theta_0}$$

$+2\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2}e^{5i\theta_0} - (4\frac{(q-r\bar{y})\bar{y}^3}{(1+\bar{y}+r\bar{y})^4} + \frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2})e^{4i\theta_0} + (2\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + 2\frac{(q-r\bar{y})^2\bar{y}}{(1+\bar{y}+r\bar{y})^3} + 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3$

$-2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2)e^{3i\theta_0} + (\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}} + 4\frac{(q-r\bar{y})\bar{y}^2}{(1+\bar{y}+r\bar{y})^3})e^{2i\theta_0} + (2\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} - 2\frac{(q-r\bar{y})\bar{y}^2}{(1+\bar{y}+r\bar{y})^3})e^{i\theta_0} + (1$

$+(\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 5(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 4\frac{(q-r\bar{y})^2\bar{y}^2}{(1+\bar{y}+r\bar{y})^4} + 4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^4) + (2\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} - 2\frac{(q-r\bar{y})\bar{y}^2}{(1+\bar{y}+r\bar{y})^3})e^{-i\theta_0}$

$+(4\frac{(q-r\bar{y})\bar{y}^2}{(1+\bar{y}+r\bar{y})^3} + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})e^{-2i\theta_0} + (2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3 - 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + 2\frac{(q-r\bar{y})^2\bar{y}}{(1+\bar{y}+r\bar{y})^3})e^{-3i\theta_0}$

$-(4\frac{(q-r\bar{y})\bar{y}^3}{(1+\bar{y}+r\bar{y})^4} + \frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2})e^{-4i\theta_0} + 2\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2}e^{-5i\theta_0} - (4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}})e^{-6i\theta_0}$

$-2\frac{(q-r\bar{y})\bar{y}^2}{(1+\bar{y}+r\bar{y})^3}e^{-7i\theta_0} - 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2e^{-9i\theta_0}]$.

The denominator of N_3 multiplied by it's conjugate is equal $(1+\bar{y}+r\bar{y})^8F_5$ where $F_5 = (1 + (\frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 5(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 4\frac{(q-r\bar{y})^2\bar{y}^2}{(1+\bar{y}+r\bar{y})^4} + 4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^4)$

$+2(2\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2} - 2\frac{(q-r\bar{y})\bar{y}^2}{(1+\bar{y}+r\bar{y})^3})\cos\theta_0 + 2(4\frac{(q-r\bar{y})\bar{y}^2}{(1+\bar{y}+r\bar{y})^3} + \frac{q-r\bar{y}}{1+\bar{y}+r\bar{y}})\cos 2\theta_0$

$+2(2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3 - 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2 + 2\frac{\bar{y}}{1+\bar{y}+r\bar{y}} + 2\frac{(q-r\bar{y})^2\bar{y}}{(1+\bar{y}+r\bar{y})^3})\cos 3\theta_0 - 2(4\frac{(q-r\bar{y})\bar{y}^3}{(1+\bar{y}+r\bar{y})^4} + \frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2})\cos 4\theta_0$

$+4\frac{(q-r\bar{y})\bar{y}}{(1+\bar{y}+r\bar{y})^2}\cos 5\theta_0 - 2(4(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^3 + \frac{\bar{y}}{1+\bar{y}+r\bar{y}})\cos 6\theta_0 - 2\frac{(q-r\bar{y})\bar{y}^2}{(1+\bar{y}+r\bar{y})^3}\cos 7\theta_0 - 2(\frac{\bar{y}}{1+\bar{y}+r\bar{y}})^2\cos 9\theta_0$

where $\cos\theta_0 = -\frac{1}{2}(\frac{q-r\bar{y}+\bar{y}}{1+\bar{y}+r\bar{y}})$,
$\cos 2\theta_0 = 2\cos^2\theta_0 - 1$,
$\cos 3\theta_0 = 3\cos^3\theta_0 - 3\cos\theta_0$,
$\cos 4\theta_0 = 8\cos^4\theta_0 - 8\cos^2\theta_0 + 1$
$\cos 5\theta_0 = 12\cos^6\theta_0 - 18\cos^3\theta_0 + 5\cos\theta_0$,
$\cos 6\theta_0 = 18\cos^6\theta_0 - 36\cos^4\theta_0 + 18\cos^2\theta_0 - 1$,
$\cos 7\theta_0 = 48\cos^8\theta_0 - 24\cos^6\theta_0 - 72\cos^5\theta_0 + 53\cos^3\theta_0 - 7\cos\theta_0$,
$\cos 9\theta_0 = 192\cos^{10}\theta_0 - 192\cos^8\theta_0 - 288\cos^7\theta_0 + 24\cos^6\theta_0 + 288\cos^5\theta_0 - 116\cos^3\theta_0 + 9\cos\theta_0$.

$$a(p^*) = \frac{1}{2}N_1 + N_2 + \frac{1}{2}N_3$$

or

$$a(p^*) = \frac{1}{2}(\frac{F_1}{(1+\bar{y}+r\bar{y})^3F_2}) + \frac{F_3}{(1+\bar{y}+r\bar{y})^2F_2} + \frac{1}{2}(\frac{F_4}{(1+\bar{y}+r\bar{y})^4F_5}).$$

Theorem 5.4. *If $a(p^*) < 0$ (respectively, > 0), then Neimark-Saker bifurcation of system (5.1.2) at $p = p^*$ is supercritical (respectively, subcritical) and there exists a unique invariant closed curve bifurcates from the positive fixed point $\bar{y}$ which is asymptotically stable (respectively, unstable).*

5.4 Numerical Discursion

In this section we give numerical examples which support our results in the previous section. Figures that we get using Matlab will be attached with example to illustrate the bifurcation.

Take the case when $q = 4$ and $r = 0.3$.

We have the equation

$$y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3y_{n-2}} \tag{5.4.1}$$

Let the initial conditions be $y_{-2} = y_{-1} = y_0 = 1$.

$$\bar{y} = \frac{3+\sqrt{9+5.2p}}{2.6}$$

Note that for $p < \frac{q}{1+r} = 3.0769$, the bifurcation point p^* is satisfy

$$\frac{1+\frac{3+\sqrt{9+5.2p^*}}{2}}{4+.7(\frac{3+\sqrt{9+5.2p^*}}{2.6})} = \frac{4-.3(\frac{3+\sqrt{9+5.2p^*}}{2.6})}{1+\frac{3+\sqrt{9+5.2p^*}}{2}}$$

$$(.25+\frac{.21}{(2.6)^2})(3+\sqrt{9+5.2p^*})^2 + (1+\frac{1.2}{2.6}-\frac{2.8}{2.6})(3+\sqrt{9+5.2p^*}) - 15 = 0$$

$$3+\sqrt{9+5.2p^*} = \frac{-(1+\frac{1.2}{2.6}-\frac{2.8}{2.6}) + \sqrt{(1+\frac{1.2}{2.6}-\frac{2.8}{2.6})^2 + 4\times 15\times(0.25+\frac{0.21}{(2.6)^2})}}{2(0.25+\frac{0.21}{(2.6)^2})}$$

$$p^* = \frac{([\frac{-(1+\frac{1.2}{2.6}-\frac{2.8}{2.6})+\sqrt{(1+\frac{1.2}{2.6}-\frac{2.8}{2.6})^2+4\times 15\times(0.25+\frac{0.21}{(2.6)^2})}}{2(0.25+\frac{0.21}{(2.6)^2})}]-3)^2-9}{5.2}$$

$$p^* = 0.83564585$$

Now we will check if Neimark-Sacker bifurcation conditions hold. By theorem (5.2), it is enough to check if

$$p^* > \frac{\left(2(1+r)\left(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\right)-(q-1)\right)^2-(q-1)^2}{4(1+r)}.$$

Note that at $p = p^*$, $\bar{y} = 2.55889613$ and

$$\frac{1+\bar{y}+r\bar{y}}{q-r\bar{y}+\bar{y}} = \frac{1+.3\times 2.55889613+2.55889613}{4-.3\times 2.55889613+2.55889613} = .74708948,$$

and

$$\frac{\left(2(1+r)\left(\frac{-(13r^2+16r-7)+\sqrt{(13r^2+16r-7)^2+4(6r-9)(9r^3+16r^2+7r)}}{2(9r^3+16r^2+7r)}\right)-(q-1)\right)^2-(q-1)^2}{4(1+r)}$$

$$= \frac{\left(2(1.3)(\frac{-(-13(0.3)^2+16\times 0.3-7)+\sqrt{(13(0.3)^2 16\times 0.3-7)^2+4(6\times 0.3-9)(9(0.3)^3+16(0.3)^2+7\times 0.3)}}{2(9(0.3)^3+16(0.3)^2+7\times 0.3)})-3\right)^2-(3)^2}{4(1.3)}$$

$$= -1.1334411 < 0.83564585 = p^*.$$

So the condition of theorem (5.2) is satisfied. That implies equation (5.4.1) undergoes a Neimark-Sacker bifurcation at $p = p^* = 0.83564585$.

The bifurcation diagram of equation (5.4.1) is shown in Figure (5.1). Figure (5.1) shows that the positive fixed point $\bar{y}$ is asymptotically stable for $p > p^*$ and change it's stability at Neimark-Sacker bifurcation value p^* and an invariant simple closed curve appears on the plane $(x(n), x(n-2))$ for $p < p^*$. Figure (5.2) and figure (5.3) shows the phase portraits associated with figure (5.2) for $p = p^*$ and $p = 0.95$, respectively.

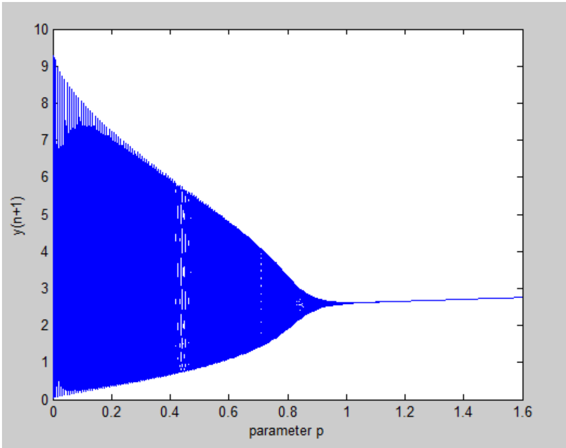

Fig. 5.1: Neimark-Sacker bifurcation of the map $y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3yn-2}$, p is a parameter.

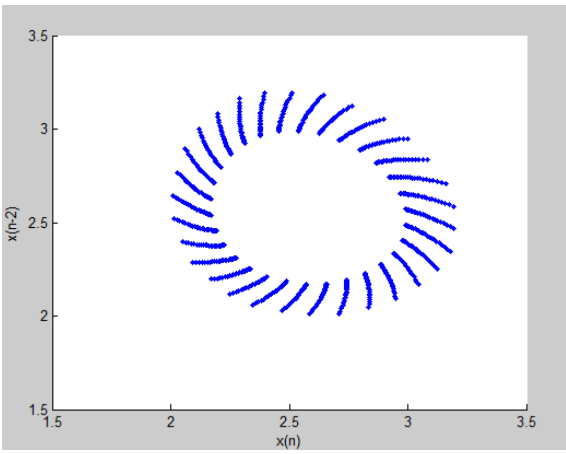

Fig. 5.2: Phase portraits of the map $y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3yn-2}$ for $p = p^*$.

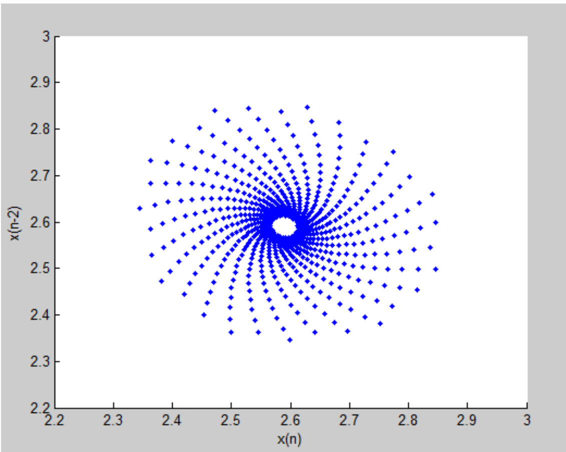

Fig. 5.3: Phase portraits of the map $y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3yn-2}$ for $p = 0.95$.

Appendices

.1 Matlab code for chapter one

figure (1.1)

The fixed points of $f(x) = 3x - x^2$

```
x0=0.2;      %initial population x0
n=200;          %end of time interval
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=x0;
tt(1)=0;
for i=1:n
t(i)=i-1;
x(i+1)=x(i)*(3-x(i));
end
t(n+1)=n;
nn=100;
del=3./nn;
xstart=0;
yy=zeros(nn+1,1);
lin=zeros(nn+1,1);
xx=zeros(nn+1,1);
for i=1:nn+1
xx(i)=xstart+(i-1)*del;
lin(i)=xx(i);
yy(i)=xx(i)*(3-xx(i));
end
plot(xx,lin,xx,yy)
title('fixed points of f(x)=x(3-x)')
```

Figure(1.2) and (1.3)

The Cobweb diagram and the behavior of the solutions near the fixed point of $f(x) = 3x - x^2$.

```
r=3;        % growth rate
x0=0.2;       %initial population x0
n=80;           %end of time interval
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=x0;
tt(1)=0;
for i=1:n
t(i)=i-1;
x(i+1)=x(i)*(r-x(i));
end
t(n+1)=n;
nn=100;
del=3./nn;
xstart=0;
yy=zeros(nn+1,1);
lin=zeros(nn+1,1);
xx=zeros(nn+1,1);
for i=1:nn+1
xx(i)=xstart+(i-1)*del;
lin(i)=xx(i);
yy(i)=xx(i)*(r-xx(i));
end
plot(xx,lin,xx,yy)
title('cobweb diagram for f(x)=x(2-x), x0=0.2'),pause
xc=zeros(24,1);
```

```
yc=zeros(24,1);
xc(1)=x0;
yc(1)=0;
xc(2)=x0;
yc(2)=x0*(r-x0);
yc(3)=yc(2);
xc(3)=yc(2);
plot(xx,lin,xx,yy,xc,yc),axis ([0 2 0 1.5]),pause
for j=3:20;
jj=2*j-4;
xc(jj)=xc(jj-1);
yc(jj)=xc(jj)*(r-xc(jj));
xc(jj+1)=yc(jj);
yc(jj+1)=yc(jj);
plot(xx,lin,xx,yy,xc,yc), axis ([0 2 0 1.5]),pause
end
plot(t,x,t,x,'.')
xlabel('n-iteration'),ylabel('x(n)'), axis ([0 80 0 1.5])
 title ('stability of fixed point')
```

.2 Matlab code for chapter two

Figure (2.1)

Saddle-node bifurcation of the map $f(x) = x^2 - cx$.

```
figure (8), hold on
for n=-0.25:0.0001:0.75
x=-(1-sqrt(1+4*n))/2;
plot(n,x,'-','MarkerSize',6)
```

```
axis([-0.5 1 -2 2])
end
for n=-0.25:0.01:0.75
y=-(1+sqrt(1+4*n))/2;
plot (n,y,'--','MarkerSize',6)
title('Saddle-node bifurcation of the map f(x)=x^2-c x')
xlabel('parameter c'),ylabel('x')
end
hold off
```

Figure(2.2)

Transcritical bifurcation of the map $f(x) = x^2 + cx$.

```
figure (6), hold on
for n=-1:0.03:1
x=1-n;
plot(n,x,'--','MarkerSize',6)
axis([-2 4 -2.5 2.5])
end
for n=-1:0.001:1
y=0;
plot (n,y,'--','MarkerSize',6)
end
for n=1:0.001:3
    x=1-n;
    plot(n,x,'--','MarkerSize',6)
end
for n=1:0.03:3
    y=0;
    plot(n,y,'--','MarkerSize',6)
```

```
    title(Transcritical bifurcation of the map f(x)=x^2+c x)
    xlabel('parameter c'),ylabel('x')
end
hold off
```

Figure(2.3)

Pitchfork bifurcation of $f(x) = cx - 2x^3$

```
figure (7), hold on
for i=-1:0.001:1
    x=0;
    plot (i,x,'MarkerSize',6), axis([-1.2 2.2 -1 1])
end
for i=1:0.001:2
    x=-sqrt((i-1)/2);
    y=sqrt((i-1)/2);
    plot (i,x,'blue',i,y,'blue','MarkerSize',6)
end
for i=1:0.04:2
    x=0;
    plot(i,x,'-', 'MarkerSize',6)
    xlabel('parameter c')
    ylabel('x')
    title(Pitchfork bifurcation of f(x)=c x -2x^3)
end
hold off
```

.3 Matlab Code For Chapter Three

Figure (3.2)

The behavior of the solutions of $y_{n+1} = \frac{4+5y_{n-1}}{1+y_n+0.5y_{n-1}}$.

```
n=70;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;
tt(1)=0;
for i=2:n
t(i)=i-1;
x(i+1)=(4+5*x(i-1))/(1+x(i)+0.5*x(i-1));
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)')
 axis ([0 70 0 10]), title ('unstabile fixed point')
```

Figure (3.1)

The behavior of the solutions of $y_{n+1} = \frac{0.8+1.5y_{n-3}}{1+y_n+0.5y_{n-3}}$.

```
n=150;
x=zeros(n+1,1);
t=zeros(n+1,1);
x(1)=0.1;x(2)=1.1;x(3)=0.2;x(4)=1;
for i=4:n
t(i)=i-1;
x(i+1)=(0.8+1.5*x(i-3))/(1+x(i)+0.5*x(i-3));
```

```
end
t(n+1)=n;
plot(t,x,t,x,'.'),xlabel('n-iteration'),ylabel('x(n)'), axis ([0 150 0 1.5])
 title ('stabile fixed point')
```

.4 Matlab Code For Chapter Four

Figure (4.1)

Period-doubling bifurcation of $y_{n+1} = \frac{1+qy_{n-1}}{1+y_n+0.09y_{n-1}}$.

```
amin=0;
amax=10;
x0=.2;x1=.3;
n=1000;
jmax=200;
t=zeros(jmax+1,1);
z=zeros(jmax+1,250);
del=(amax-amin)/jmax;
for j=1:jmax+1
x=zeros(n+1,1);
x(1)=x0;x(2)=x1;
t(j)=(j-1)*del+amin;
a=t(j);
for i=2:n
x(i+1)=(1+a.*x(i-1))/(1+x(i)+.9.*x(i-1));
if (i>750)
   z(j,i-750)=x(i+1);
   end
end
```

```
end
plot(t,z,'blue','MarkerSize',5),title ('Period-doubling bifurcation')
```

Figure (4.2)

Period-doubling bifurcation of $y_{n+1} = \frac{p+1.1y_{n-1}}{1+y_n+0.09y_{n-1}}$

```
amin=0;
amax=2;
x0=.2;x1=.3;
n=1000;
jmax=200;
t=zeros(jmax+1,1);
z=zeros(jmax+1,250);
del=(amax-amin)/jmax;
for j=1:jmax+1
x=zeros(n+1,1);
x(1)=x0;x(2)=x1;
t(j)=(j-1)*del+amin;
a=t(j);
for i=2:n
x(i+1)=(a+1.1.*x(i-1))/(1+x(i)+.09.*x(i-1));
if (i>750)
   z(j,i-750)=x(i+1);
   end
end
end
plot(t,z,'blue','MarkerSize',6)
title ('Period-doubling bifurcation'),xlabel('parameter q'), ylabel('x(n)')
```

.5 Matlab Code For Chapter Five

Figure (5.1)

Neimark-Sacker bifurcation of $y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3y_{n-2}}$.

```
amin=0;
amax=1.6;
x0=1;x1=1;
n=1000;
jmax=200;
t=zeros(jmax+1,1);
z=zeros(jmax+1,250);
del=(amax-amin)/jmax;
for j=1:jmax+1
x=zeros(n+1,1);
x(1)=x0;x(2)=x1;x(3)=1;
t(j)=(j-1)*del+amin;
a=t(j);
for i=3:n
x(i+1)=(a+4.*x(i-2))/(1+x(i)+.3.*x(i-2));
if (i>750)
   z(j,i-750)=x(i+1);
   end
end
end
plot(t,z,'blue','MarkerSize',6),xlabel('parameter p'), ylabel('y(n+1)')
```

Matlab code of figure (5.2)

Phase portraits diagram of $y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3y_{n-2}}$ $p = p^*$.

```
$N=1000; x(1)=1;x(2)=1;x(3)=1;
for p=0.8354585
    for n=3:1:0.3*N
        x(n+1)=(p+4*x(n-2))/(1+x(n)+0.3*x(n-2));
        x(n-1);
    end
    figure (2), hold on
    for n=0.3*N :1:N
        x(n+1)=(p+4*x(n-2))/(1+x(n)+0.3*x(n-2));
        x(n);
        plot(x(n),x(n-2),'.','MarkerSize',5),axis([2.2 3.2 2.2 3.2])
        xlabel('x(n)'),ylabel('x(n-2)')
    end
end
hold off
```

Matlab code of figure (5.3)

Phase portraits diagram of $y_{n+1} = \frac{p+4y_{n-2}}{1+y_n+0.3y_{n-2}}$ $p = 0.95$.

```
N=1000; x(1)=1;x(2)=1;x(3)=1;
for p=0.95
    for n=3:1:0.3*N
        x(n+1)=(p+4*x(n-2))/(1+x(n)+0.3*x(n-2));
        x(n-1);
    end
```

```
    figure (2), hold on
    for n=0.3*N :1:N
        x(n+1)=(p+4*x(n-2))/(1+x(n)+0.3*x(n-2));
        x(n);
        plot(x(n),x(n-2),'.','MarkerSize',5),axis([2.2 3 2.2 3])
        xlabel('x(n)'),ylabel('x(n-2)')
    end
end
hold off
```

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